Representation and duality of the untyped λ-calculus in nominal lattice and topological semantics, with a proof of topological completeness

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We give a semantics for the λ-calculus based on a topological duality theorem in nominal sets. A novel interpretation of λ is given in terms of adjoints, and λ-terms are interpreted absolutely as sets (no valuation is necessary).

Additional Key Words and Phrases: Nominal algebras, fresh-finite limits, lambda-calculus, spectral spaces, lattices and order, variables, nominal techniques, mathematical foundations, Fraenkel-Mostowski set theory

Contents

1. INTRODUCTION

In this paper we build a topological duality result for the untyped λ-calculus in nominal sets and prove soundness, completeness, and topological completeness.

This means the following:

— We give a lattice-style axiomatisation of the untyped λ-calculus and prove it sound and complete.
— We define a notion of topological space whose compact open sets have notions of application and λ-abstraction.
— We prove that the categories of lattices-with-λ and topological-spaces-with-λ are dual.
— We give a complete topological semantics for the λ-calculus.

So this paper does what Stone duality does for Boolean algebras [Joh86], but for the λ-calculus.

1.1. A very brief summary of the contributions

We summarise some contributions of the paper; this list will be fleshed out in the rest of the Introduction:

(1) No previous duality result for λ-calculus theories exists. Duality results are interesting in themselves (see next subsection), and it is interesting to see how nominal techniques help to manage the technical demands of such a result.
(2) The topological representations obtained are concrete, being based on nominal sets.
(3) The representation of open terms does not use valuations; possibly open λ-terms are interpreted as open sets in a nominal topological space (this is sometimes called an *absolute* semantics). Function application and also λ-abstraction get interpreted as concrete sets operations on nominal-style atoms.
(4) This paper is a nontrivial application of nominal ideas, and the techniques on nominal sets which we use are original and have independent technical interest.[1]
(5) We make nominal-style atoms—urelemente in Fraenkel-Mostowski set theory—behave like variables of the λ-calculus. Urelemente in set theory come equipped with very few properties; indeed, by design urelemente have virtually no properties at all. It is remarkable that they can nevertheless acquire such rich structure.[2]
(6) β-reduction and η-expansion are exhibited as adjoint properties.
(7) The fine structure of the canonical models (those of the form $points_{\Pi}$) is very rich, as we shall see. The use of canonical models in this paper probably does not exhaust their interest.
(8) We prove a topological completeness result—but this should be impossible: topological *in*completeness results exist in the literature.
This depends on the topology, so the fact that a notion of topology that 'works' for the λ-calculus exists, is surprising given the current state of the art. One would not expect this to work.

1.2. The point of duality results

What is the point of a duality result, and why bother doing it for the λ-calculus?

(1) Duality results are a strong form of completeness: for a given class of abstract models, every model has a concrete topological representation (i.e. in terms of sets with a few consistency conditions) and every map between models has a concrete representation as a continuous map (i.e. a map that has to respect those conditions).

[1]It may be worth amplifying on this, just a little. We face design questions such as: *Should the points be finitely-supported, assuming the open sets are? What is the proper notion of filter, in this context? Should covering sets be finitely supported? Should they be strictly finitely supported? In the presence of an* ʖ*-action, what is the correct notion of freshness for a set of points, anyway?* And so on; dozens of such questions are addressed in the technical material to follow.

Anybody doing topology and/or duality in nominal sets—we hope and expect that others will follow us here—will encounter similar questions, and so could benefit from an analysis of the answers we arrived at in this paper and in [Gab11b].

[2]We carry out a similar programme for variables of first-order logic in [Gab11b; Gab12].

Intuitively, Boolean algebras, Heyting algebras, and distributive lattices look like they all have to do with powersets (negation is some kind of sets complement, conjunction is sets intersection, disjunction is sets union, and so on). But is this true? Is it possible to construct some sufficiently bizzare model such that for instance conjunction *must* mean something other than sets intersection? The answer is no: duality theorems tell us that no matter how bizzare the model, it *can* be represented topologically. In topologies conjunction is sets intersection.
The same analysis is applicable to our duality result for the λ-calculus.

(2) Of course, it is not obvious how conjunction enters into the untyped λ-calculus. Indeed, no duality result has been achieved for the λ-calculus before and it is not obvious even how to begin to go about this.
One contribution of this paper is that we embed the λ-calculus in an impredicative logic which we characterise in two ways: in nominal algebra, and using a nominal generalisation of finite limits which we call *fresh-finite limits* (Definition 4.1.2).
An example observation that comes out of this is that we exhibit β-reduction and η-expansion as adjoint maps (counit and unit respectively; see Proposition 10.2.4).
Another example is that λ features in this paper as a derived object made out of the fresh-finite limit $\forall$, and a right adjoint to application $\multimap\!\bullet$. See Notation 10.2.1 and the subsequent discussion. The quantifier $\forall$ is itself an interesting entity, a kind of arbitrary conjunction, which relies heavily on nominal techniques. More on this later.
Thus, in the process of defining and proving our results—representation, duality, and completeness—we uncover a wealth of structure in the untyped λ-calculus (to add to the wealth of structure already known). The technical definitions and lemmas which our 'main results' depend on, are as interesting as the results themselves.

(3) Finally, we note that our topological semantics for the λ-calculus is *complete* (Theorem 11.9.5). This is remarkable because Salibra has shown that all known semantics for the λ-calculus based on partial orders, are incomplete [Sal03]. The fact that our semantics is topological (thus ordered) and complete, is unexpected.[3] We discuss this apparent paradox in Subsection 12.1.5.
In any case, new semantics for the untyped λ-calculus do not come along very often, and as mentioned above, no duality result for the λ-calculus has been proved before.

More interesting structure will be uncovered by this way of approaching the λ-calculus; the list above justifies why it is *a priori* interesting to try.

We would like to address one more obvious question: why is this paper so long?

Even granted that the results are interesting, reading them may require some stamina.[4] Does it have to be this way? We think this is reasonable, because:

(1) This paper takes on two notoriously challenging types of proof: a duality result (for a complex logic) *and* a semantics for and completeness proof for the untyped λ-calculus.
It is simply a fact that duality results are hard, completeness proofs are also hard, and semantics for the untyped λ-calculus are not trivial to construct.[5]

(2) This paper is based on nominal techniques. This is a relatively less developed and less familiar environment than the well-trodden Zermelo-Fraenkel/higher-order logic framework, so we have to build our tools as we go along. Even when material is taken from previous work, we cannot assume the reader is familiar with it.
In short, there can be relatively less hand-waving. Where we can rely on the material being familiar, we will be more brief.

[3] Our paper handles only the case where we have η-expansion. We believe this could be generalised to the fully non-extensional case, at some cost in complexity. See Subsection 12.1.3.

[4] Writing them certainly did.

[5] There are actually two duality results: one for impredicative distributive lattices (corresponding to a propositional logic with $\wedge$, $\vee$, and propositional quantification) from Part I, and the other when we add the combination structure in Part III. The second duality piggy-backs on the first, and is shorter.

Given that we undertake two proofs which are known to be meaty, about a system whose semantics are challenging, and given that we build the tools for this from first principles, it may be more surprising that the paper is not longer.

So our starting point is urelemente (atoms) of Fraenkel-Mostowski set theory; our target is the untyped λ-calculus, and between them there is a lot of ground to cover. Everything in this paper is there because its has to be, it is worth the effort, and the material has natural momentum which propels us from the first definitions to the final results.

1.3. Background motivation, in a nutshell

Interpretations of $\forall x.\phi(x)$ or $\lambda x.f(x)$ usually involve an explicit quantification over some domain. This is reflected in semantics, which involve a *valuation* assigning interpretations to variables.

Alternatively we can interpret variables as atoms in Fraenkel-Mostowksi sets. Originally this was applied to model inductive syntax [Gab01; GP01]. This paper is part of a research programme to apply the idea to models of logic and computation.

The key definitions go back to [GM06a; GM06b], where nominal algebraic notions of substitution and first-order logic were given. Variables receive a fixed interpretation in the model, as atoms. Instead of valuations is a nominal algebraic generalisation of substitution—this is the *σ-action* of this paper.

The payoff is interesting new classes of models. There is quite an extensive literature on this, discussed in the Conclusions. Most recently, in [Gab11b; Gab12] we consider nominal lattice and topological models of first-order logic—and in this paper we do something similar for the λ-calculus, though this is harder since the λ-calculus is more complex.

Universal quantification $\forall$ becomes a special kind of limit called a *fresh-finite limit*. On a σ-algebra this happens to coincide with an infinite conjunction of substitution instances, but that is a theorem, not a definition. λ splits into two halves, and essentially becomes a corollary of $\forall$ and application.

Using this we obtain axiomatic, lattice-theoretic, topological, and concrete models of the λ-calculus. We are used to seeing such things for the propositional case, for instance for Boolean algebras. It is more unusual to see this applied to languages with binders, but that is what nominal techniques help us to deal with. In doing this, we also give a complete topological semantics for the λ-calculus.

1.4. Map of the paper

Section 2 sets up some basic nominal theory. The reader might like to skim this at first, since the definitions might only make sense in terms of their application later on in the paper. Highlights are the notions of nominal set (Definition 2.1.5), finitely-supported and strictly-finitely supported powersets (Subsections 2.4.1 and 2.4.2), equivariance properties of atoms and the И-quantifier (Theorem 2.3.1 and Definition 2.3.6), and the И-quantifier for sets (Subsection 2.5).

Section 3 introduces σ- and ᘔ-algebras. These are the basic building blocks from which our models will be constructed. The definitions are already non-trivial; highlights are the axioms of Figure 1 (which go back to [GM06a], where nominal algebra was introduced to axiomatise substitution) and [Gab11b] (which introduced ᘔ-algebras), and the precise definition of the σ-action on nominal powersets in Definition 3.4.1, which uses the И-quantifier.

Section 4 considers lattices over nominal sets. *The* technical highlight here is the characterisation of universal quantification in terms of *fresh-finite limits* (Definition 4.1.1 and subsequent results). Combined with impredicativity and the σ-action we arrive at Definitions 4.5.1 and 4.5.9, which are the lattice-theoretic structure within which we eventually build models of the λ-calculus which we write $\mathsf{inDi}\forall$ (pronounced 'India').

Section 5 shows that (simplifying) every nominal powerset is a model of Definition 4.5.1. A technical highlight here is Proposition 5.2.8: a characterisation of universal quantification on nominal sets in terms of universal and new-quantification over all atoms. Definition 5.2.1 is also interesting just because it shows how the σ-action can be combined with other sets operations to do predicate logic.

Section 6 uses filters and prime filters to give a nominal sets representation of any $\mathsf{inDi}\forall$, and Section 7 extends this to a full duality.

This duality is for a propositional logic with quantifiers. To handle the λ-calculus we need more: this happens in Section 9. The most important points are probably the introduction of $\bullet$ and its right adjoint $-\!\!\bullet$ in Definition 9.1.1, and the observation that their topological dual is a *combination operator* $\circ$ in Definition 9.2.1.

It now becomes fairly easy to show that every $\mathsf{inDi}\forall_{\bullet}$ is also a semantics for the untyped λ-calculus. This is Section 10, culminating in Definition 10.4.1 and Theorem 10.4.7; we include an interlude in Subsection 10.5 where we pause to take stock of what we have been doing so far.

Slightly harder is proving completeness, for which we must construct an object in $\mathsf{inDi}\forall_{\bullet}/\mathsf{inSpect}\forall_{\bullet}$ for any given λ-theory. This is Section 11: we have the tools (nominal and otherwise) required in principle to carry out the constructions (just build filters, etcetera)—but in practice the amount of detailed structure required to make this work is quite striking, involving amongst many other things the construction of $\bullet$ and $-\!\!\bullet$ on points (Definition 11.3.1) and a left adjoint to the $\mathfrak{r}$-algebra on filters (Definition 11.4.1). If two results should illustrate how tightly knit this part of the mathematics can be, then the technical results of Proposition 11.1.6 and Lemma 11.4.9 are good examples. The final Completeness result is Theorem 11.9.5.

We conclude in an Appendix with a nominal axiomatisation of $\forall$, to go with the lattice-theoretic one of Definition 4.1.2, and some nice additional observations on the structure of points.

1.5. A list of interesting technical definitions

This list is not of the major results, nor is it an exhaustive list of technical definitions. But one technical definition or proof looks very much like another, so here are suggestions of which technical highlights might be worth looking at first:

— We characterise universal quantification in four different ways, all of which have their place in the paper: as a fresh-finite limit in Definition 4.1.2, using quanitification over all atoms in Proposition 5.2.8, slightly indirectly using the И-quantifier in Definition 6.1.1, and finally using nominal algebra axioms in Appendix A.1.
— As mentioned above, β-reduction and η-expansion are derived from a counit and unit respectively in Proposition 10.2.4.
— We decompose of λ into $\forall$ and $-\!\!\bullet$ in Notation 10.2.1. We decompose $\forall$ further into atoms-quantification in Proposition 5.2.8.
— Proposition 5.2.8 and Lemma 11.5.2 are inherently surprising results.
— Three little proof gems are in Proposition 11.1.8, Lemma 11.2.4, and Lemma 11.5.1.
— We need two notions of filter: one is Definition 6.1.1 (the interesting new part is condition 4), the other is Definition 11.1.3 (we call it a *point*). Both notions use the И-quantifier in interesting ways.
— The pointwise definition of substitution is in Definition 3.4.1; a different approach to substitution than the reader has likely seen, which makes use of the И-quantifier and $\mathfrak{r}$-algebra (Definition 3.2.1). The two concrete characterisations of it in Subsection 11.4.2 are also interesting.
— The reader will see much use made of the nominal И-quantifier, meaning 'for all but finitely many atoms', and of nominal equivariance properties and notions of finite support and *strict* finite support. We mention two (connected) examples: the use of И in defining the σ-action in Definition 3.4.1, and the treatments of the universal quantifier in Proposition 5.2.8 and in condition 4 of Definition 6.1.1. A search of the paper for uses of Corollary 2.1.10, Theorems 2.3.1 and 2.3.9, and Lemma 2.4.3 will find many more.

2. BACKGROUND ON NOMINAL TECHNIQUES

A nominal set is a 'set with names'. The notion of a name being 'in' an element is given by support $supp(x)$ (Definition 2.1.7). For more details of nominal sets, see [GP01; Gab11a].

Here we just give necessary background information. The reader not interested in nominal techniques *per se* might like to read this section only briefly in the first instance, and use it as a reference for the later sections, where the ideas get applied.

For the reader's convenience we take a moment to note the overall message of this section:

— *To the category-theorist* we say that we work mostly in the category of nominal sets, or equivalently in the Schanuel topos (more on this in [MM92, Section III.9], [Joh03, A.21, page 79], or [Gab11a, Theorem 9.14]), and occasionally also in the category of sets with a permutation action.
— *To the set-theorist* we say that our constructions can be carried out in Fraenkel-Mostowski set theory (**FM sets**) and Zermelo-Fraenkel set theory with atoms (**ZFA**). A discussion of such sets foundations, tailored to nominal techniques, can be found in [Gab11a, Section 10]).
— *To the reader not interested in foundations* we say that the apparently inconsequential step of assuming names as primitive entities in Definition 2.1.1 gives us a remarkable clutch of definitions and results, notably Theorem 2.1.9 and Corollary 2.1.10, and Theorems 2.3.1 and 2.3.9. These properties are phrased abstractly but will quickly make themselves very useful in the body of this paper. See previous work for more background [GP01; Gab11a; Gab13].

2.1. Basic definitions

DEFINITION 2.1.1. Fix a countably infinite set of **atoms** $\mathbb{A}$. We use a **permutative convention** that $a, b, c, \dots$ range over *distinct* atoms.

DEFINITION 2.1.2. A **(finite) permutation** π is a bijection on atoms such that $nontriv(\pi) = \{a \mid \pi(a) \neq a\}$ is finite.

Write id for the **identity** permutation such that $\mathrm{id}(a) = a$ for all a. Write $\pi' \circ \pi$ for composition, so that $(\pi' \circ \pi)(a) = \pi'(\pi(a))$. Write $\pi^{\text{-}1}$ for inverse, so that $\pi^{\text{-}1} \circ \pi = \mathrm{id} = \pi \circ \pi^{\text{-}1}$. Write $(a\ b)$ for the **swapping** (terminology from [GP01]) mapping a to b, b to a, and all other c to themselves, and take $(a\ a) = \mathrm{id}$.

NOTATION 2.1.3. If $A \subseteq \mathbb{A}$ write

$$fix(A) = \{\pi \mid \forall a{\in}A.\pi(a) = a\}.$$

DEFINITION 2.1.4. A **set with a permutation action** $\mathcal{X}$ is a pair $(|\mathcal{X}|, \cdot)$ of an **underlying set** $|\mathcal{X}|$ and a **permutation action** written $\pi{\cdot}_{\mathcal{X}}x$ or just $\pi{\cdot}x$ which is a group action on $|\mathcal{X}|$, so that $\mathrm{id}{\cdot}x = x$ and $\pi{\cdot}(\pi'{\cdot}x) = (\pi \circ \pi'){\cdot}x$ for all $x \in |\mathcal{X}|$ and permutations π and π'.
Say that $A \subseteq \mathbb{A}$ **supports** $x \in |\mathcal{X}|$ when $\forall \pi.\pi \in fix(A) \Rightarrow \pi{\cdot}x = x$. If a finite A supporting x exists, call x **finitely supported**.

DEFINITION 2.1.5. Call a set with a permutation action $\mathcal{X}$ a **nominal set** when every $x \in |\mathcal{X}|$ has finite support. $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ will range over nominal sets.

DEFINITION 2.1.6. Call a function f from $|\mathcal{X}|$ to $|\mathcal{Y}|$ **equivariant** when $\pi{\cdot}(f(x)) = f(\pi{\cdot}x)$ for all permutations π and $x \in |\mathcal{X}|$. In this case write $f : \mathcal{X} \to \mathcal{Y}$.

The category of nominal sets and equivariant functions between them is usually called the category of *nominal sets*.

DEFINITION 2.1.7. Suppose $\mathcal{X}$ is a nominal set and $x \in |\mathcal{X}|$. Define the **support** of x by

$$supp(x) = \bigcap \{A \mid A \text{ finite and supports } x\}.$$

NOTATION 2.1.8. Write $a\#x$ as shorthand for $a \notin supp(x)$ and read this as a is **fresh for** x.

Given atoms $a_1, \dots, a_n$ and elements $x_1, \dots, x_m$ write $a_1, \dots, a_n\#x_1, \dots, x_m$ as shorthand for $\{a_1, \dots, a_n\} \cap \bigcup_{1 \leq j \leq m} supp(x_j) = \varnothing$, or to put it more plainly: $a_i\#x_j$ for every i and j.

THEOREM 2.1.9. *Suppose $\mathcal{X}$ is a nominal set and $x \in |\mathcal{X}|$. Then $supp(x)$ is the unique least finite set of atoms that supports x.*

Proof. See part 1 of Theorem 2.21 of [Gab11a]. □

COROLLARY 2.1.10. (1) *If* $\pi(a) = a$ *for all* $a \in supp(x)$ *then* $\pi{\cdot}x = x$.
(2) *If* $\pi(a) = \pi'(a)$ *for every* $a{\in}supp(x)$ *then* $\pi{\cdot}x = \pi'{\cdot}x$.
(3) $a\#x$ *if and only if* $\exists b.(b\#x \wedge (b\ a){\cdot}x = x)$.

Proof. See part 2 of Theorem 2.21 of [Gab11a]. □

2.2. Examples

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are nominal sets, and suppose $\mathcal{Z}$ is a set with a permutation action. We consider some examples of sets with a permutation action and of nominal sets. These will be useful later on in the paper.

2.2.1. Atoms and booleans. $\mathbb{A}$ is a nominal set with the *natural permutation action* $\pi{\cdot}a = \pi(a)$.

For the case of $\mathbb{A}$ only we will be lax about the difference between $\mathbb{A}$ (the set of atoms) and $(|\mathbb{A}|, \cdot)$ (the nominal set of atoms with its natural permutation action). What that means in practice is that we will write $a \in \mathbb{A}$ and never write $a \in |\mathbb{A}|$.[6]

The only equivariant function from $\mathbb{A}$ to itself (Definition 2.1.6) is the identity map $a \mapsto a$. There are more finitely supported maps from $\mathbb{A}$ to itself; see the *finitely supported function space* below.

Write $\mathbb{B}$ for the nominal set of **Booleans**, which has elements $\{\bot, \top\}$ and the **trivial** permutation action that $\pi{\cdot}x = x$ for all π and $x \in |\mathbb{B}|$.

2.2.2. Cartesian product. $\mathcal{X} \times \mathcal{Y}$ is a nominal set with underlying set $\{(x, y) \mid x \in |\mathcal{X}|, y \in |\mathcal{Y}|\}$ and the *pointwise* action $\pi{\cdot}(x, y) = (\pi{\cdot}x, \pi{\cdot}y)$.

An equivariant $f : (\mathcal{X} \times \mathcal{Y}) \to \mathbb{B}$ corresponds to a relation $\mathcal{R}$ such that $x\ \mathcal{R}\ y$ if and only if $\pi{\cdot}x\ \mathcal{R}\ \pi{\cdot}y$.

2.2.3. Tensor product. $\mathcal{X} \otimes \mathcal{Y}$ is a nominal set with underlying set $\{(x, y) \mid x \in |\mathcal{X}|, y \in |\mathcal{Y}|,\ supp(x){\cap}supp(y) = \varnothing\}$ and the pointwise action. For the pointwise action here to be well-defined depends on π being a permutation and the fact (Proposition 2.3.3 below) that $supp(\pi{\cdot}x) = \pi{\cdot}supp(x)$.

2.2.4. Full function space. Functions from $|\mathcal{X}|$ to $|\mathcal{Y}|$ form a set with a permutation action with underlying set all functions from $|\mathcal{X}|$ to $|\mathcal{Y}|$, and the **conjugation** permutation action

$$(\pi{\cdot}f)(x) = \pi{\cdot}(f(\pi^{-1}(x))).$$

2.2.5. Finitely supported function space. $\mathcal{X}{\Rightarrow}\mathcal{Y}$ is a nominal set with underlying set the functions from $|\mathcal{X}|$ to $|\mathcal{Y}|$ with finite support under the conjugation action, and the conjugation permutation action.

A complete description of the finitely supported functions from $\mathbb{A}$ to $\mathbb{B}$ is as follows:

— The identity, mapping a to itself.
— Any function f such that there exists some $a \in \mathbb{A}$ and some finite $U \subseteq \mathbb{A}$ such that if $x \notin U$ then $f(x) = a$ (so f is 'eventually constant').

2.2.6. Full powerset

DEFINITION 2.2.1. Suppose $\mathcal{X}$ is a set with a permutation action. Give subsets $X \subseteq |\mathcal{X}|$ the **pointwise** permutation action

$$\pi{\cdot}X = \{\pi{\cdot}x \mid x \in X\}.$$

Then $powerset(\mathcal{X})$ (the full powerset of $\mathcal{X}$) is a set with a permutation action with

— underlying set $\{X \mid X \subseteq |\mathcal{X}|\}$ (the set of all subsets of $|\mathcal{X}|$, and
— the pointwise action $\pi{\cdot}X = \{\pi{\cdot}x \mid x \in X\}$.

[6] Just sometimes, pedantry has its limit.

A particularly useful instance of the pointwise action is for sets of atoms. As discussed in Subsection 2.2.1 above, if $a \in \mathbb{A}$ then $\pi \cdot a = \pi(a)$. Thus if $A \subseteq \mathbb{A}$ then

$$\pi \cdot A \quad \text{means} \quad \{\pi(a) \mid a \in A\}.$$

LEMMA 2.2.2. *Even if $\mathcal{X}$ is a nominal set, $powerset(\mathcal{X})$ need not be a nominal set. That is, $powerset(\mathcal{X})$ is not necessarily a nominal set.*

Proof. Take $\mathcal{X}$ to be equal to $\mathbb{A} = \{a, b, c, d, e, f, \dots\}$ and consider the set

$$comb = \{a, c, e, \dots\}$$

of 'every other atom'. This does not have finite support, though permutations still act on it pointwise. For more discussion of this point, see [Gab11a, Remark 2.18]. □

We consider further examples in Subsection 2.4, including the finitely-supported and strictly finitely-supported powersets.

2.3. The principle of equivariance and the И quantifier

We come to Theorem 2.3.1, a result which is central to the 'look and feel' of nominal techniques. It enables a particularly efficient management of renaming and α-conversion in syntax and semantics and captures why it is so useful to use *names* in the foundations of our semantics and not some other infinite, set such as numbers.

Names are by definition symmetric (i.e. can be permuted). Taking names and permutations as *primitive* implies that permutations propagate to the things we build using them. This is the *principle of equivariance* (Theorem 2.3.1 below; see also [Gab11a, Subsection 4.2] and [GP01, Lemma 4.7]).

The principle of equivariance implies that, provided we permute names uniformly in all the parameters of our definitions and theorems, we then get another valid set of definitions and theorems. This is not true of e.g. numbers because our mathematical foundation equips numbers by construction with numerical properties such as *less than or equal to* $\leq$, which can be defined from first principles with no parameters.

So if we use numbers for names then we do not care about $\leq$ because we just needed a countable set of elements, but we repeatedly have to *prove* that we did not use an asymmetric property like $\leq$. In contrast, with nominal foundations and atoms, we do not have to explicitly prove symmetry because we can just look at our mathematical foundation and note that it is naturally symmetric under permuting names; we reserve numbers for naturally *a*symmetric activities, such as counting.

This style of name management is characteristic of nominal techniques. The reader can find it applied often, e.g. in Lemmas 3.2.5 and 3.4.8, Propositions 3.3.4 and 4.3.5, and Lemma 5.1.2.

THEOREM 2.3.1. *If $\overline{x}$ is a list $x_1, \dots, x_n$, write $\pi \cdot \overline{x}$ for $\pi \cdot x_1, \dots, \pi \cdot x_n$. Suppose $\phi(\overline{x})$ is a first-order logic predicate with free variables $\overline{x}$. Suppose $\chi(\overline{x})$ is a function specified using a first-order predicate with free variables $\overline{x}$. Then we have the following principles:*

(*1*) **Equivariance of predicates.** $\phi(\overline{x}) \Leftrightarrow \phi(\pi \cdot \overline{x})$.[7]
(*2*) **Equivariance of functions.** $\pi \cdot \chi(\overline{x}) = \chi(\pi \cdot \overline{x})$ *(cf. Definition 2.1.6).*[8]
(*3*) **Conservation of support.** *If $\overline{x}$ denotes elements with finite support then $supp(\chi(\overline{x})) \subseteq supp(x_1) \cup \dots \cup supp(x_n)$.*
If in addition χ is injective, then $supp(\chi(\overline{x})) = supp(x_1) \cup \dots \cup supp(x_n)$.

Proof. See Theorem 4.4, Corollary 4.6, and Theorem 4.7 from [Gab11a]. □

[7]Here $\overline{x}$ is understood to contain *all* the variables mentioned in the predicate. It is not the case that $a = a$ if and only if $a = b$—but it is the case that $a = b$ if and only if $b = a$.

[8]Parts 1 and 2 of Theorem 2.3.1 are morally the same result: by considering ϕ to be a function from its arguments to the nominal set of Booleans $\mathbb{B}$ from Subsection 2.2; and by treating a function as a functional relation, i.e. as a binary predicate.

Remark 2.3.2. Theorem 2.3.1 is three fancy ways of observing that if a specification is symmetric in atoms, the the result must be at least as symmetric as the inputs. The benefit of using atoms (instead of e.g. numbers) to model names makes this a one-line argument.[9]

Proposition 2.3.3. *$supp(\pi{\cdot}x) = \pi{\cdot}supp(x)$ (which means $\{\pi(a) \mid a \in supp(x)\}$).*

Proof. Immediate consequence of part 2 of Theorem 2.3.1.[10] □

Lemma 2.3.4 goes back to [GM07, Lemma 5.2] and [GM09, Corollary 4.30]; see [Gab13, Lemma 7.6.2] for a recent presentation:

Lemma 2.3.4. *If χ is an equivariant function from $\mathcal{X}$ to $\mathcal{Y}$ and $a\#\chi(x)$ then there exists some $x' \in |\mathcal{X}|$ such that $a\#x'$ and $\chi(x) = \chi(x')$.*

Proof. Choose fresh b (so $b\#x$). By Corollary 2.1.10 $(b\ a){\cdot}\chi(x) = \chi(x)$ and by Definition 2.1.6 $(b\ a){\cdot}\chi(x) = \chi((b\ a){\cdot}x)$. We take $x' = (b\ a){\cdot}x$. □

Lemma 2.3.5. *Suppose $F : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is an equivariant function and $x \in |\mathcal{X}|$ and $y \in |\mathcal{Y}|$. Suppose further that $a, b\#y$. Then $F(x, (b\ a){\cdot}y) = (b\ a){\cdot}F(x, y)$.*

Proof. By equivariance $(b\ a){\cdot}F(x, y) = F((b\ a){\cdot}x, (b\ a){\cdot}y)$. By Corollary 2.1.10 since $a, b\#y$ also $(b\ a){\cdot}y = y$. □

Definition 2.3.6. Write $\text{И}a.\phi(a)$ for '$\{a \mid \neg\phi(a)\}$ is finite'. We call this the И **quantifier**.

Remark 2.3.7. We can read И as 'for all but finitely many a', 'for fresh a', or 'for new a'. It captures a *generative* aspect of names, that for any x we can find plenty of atoms a such that $a \notin supp(x)$. И was designed in [GP01] to model the quantifier being used when we informally write "rename x in $\lambda x.t$ to be fresh", or "emit a fresh channel name" or "generate a fresh memory cell".

Remark 2.3.8. И simply means 'for all but finitely many atoms'; it belongs to a family of 'for most' quantifiers [Wes89], and is a *generalised quantifier* [KW96, Section 1.2.1].

Specifically over nominal sets, however, И displays special properties. In particular, it satisfies the *some/any property* that to prove a И-quantified property we test it for *one* fresh atom; we may then use it for *any* fresh atom. This is Theorem 2.3.9:

Theorem 2.3.9. *Suppose $\phi(\overline{z}, a)$ is a predicate with free variables $\overline{z}, a$.[11] Suppose $\overline{z}$ denotes elements with finite support. Then the following are equivalent:*

$$\forall a.(a \in \mathbb{A} \wedge a\#\overline{z}) \Rightarrow \phi(\overline{z}, a) \qquad \text{И}a.\phi(\overline{z}, a) \qquad \exists a.a \in \mathbb{A} \wedge a\#\overline{z} \wedge \phi(\overline{z}, a)$$

Proof. See Theorem 6.5 from [Gab11a] or Proposition 4.10 from [GP01]. □

2.4. Further examples

We now consider the finitely supported powerset and the strictly finitely supported powerset. These examples are more technically challenging and will be key to the later constructions.

[9] The reasoning in this paper could in principle be fully formalised in a sets foundation with atoms, such as Zermelo-Fraenkel set theory with atoms **ZFA**. Nominal sets can be implemented in ZFA sets such that nominal sets map to equivariant elements (elements with empty support) and the permutation action maps to 'real' permutation of atoms in the model. See [Gab11a, Subsection 9.3] and [Gab11a, Section 4].

[10] There is also a nice proof of this fact by direct calculations; see [Gab11a, Theorem 2.19]. However, it just instantiates Theorem 2.3.1 to the particular χ specifying support.

[11] ϕ should not use the axiom of choice. Every ϕ used in this paper will satisfy this property.

2.4.1. Finitely supported powerset. Suppose $\mathcal{X}$ is a set with a permutation action (it does not have to be a nominal set).

Then $pow(\mathcal{X})$, the **nominal powerset**, is a nominal set, with

— underlying set those $X \in |powerset(\mathcal{X})|$ that are finitely supported, and
— with the **pointwise** action $\pi{\cdot}X = \{\pi{\cdot}x \mid x \in X\}$ inherited from Definition 2.2.1.

Unpacking the definitions and using Corollary 2.1.10, $X \subseteq |\mathcal{X}|$ is finitely supported when, equivalently:

— There exists some finite $A \subseteq \mathbb{A}$ such that if $\pi \in \mathit{fix}(A)$ then $\pi{\cdot}X = X$.
— There exists some finite $A \subseteq \mathbb{A}$ such that if $\pi \in \mathit{fix}(A)$ and $x \in X$ then $\pi{\cdot}x \in X$.
— $\text{И}a.\text{И}b.(a\ b){\cdot}X = X$.
— $\text{И}a.\text{И}b.\forall x.(x{\in}X \Rightarrow (a\ b){\cdot}x{\in}X)$.

For instance:

— $pow(\mathbb{A})$ is the set of finite and cofinite sets of atoms (a set of atoms is cofinite when its complement is finite).
— $X \in pow(powerset(\mathbb{A}))$ is a set of sets of atoms with finite support, though the elements $x \in X$ need not have finite support.
For instance, if we set $x = comb$ from Lemma 2.2.2 then we can take $X = \{\pi{\cdot}x \mid \text{all permutations } \pi\}$. Here X has finite (indeed, empty) support, even though none of its elements $\pi{\cdot}x$ have finite support.

It is useful to formalise these observations as a lemma. A common source of confusion is to suppose that if A supports $X \in |pow(\mathcal{X})|$ then A must supports every $x \in X$. This is incorrect:

Lemma 2.4.1. *It is not true in general that if $X \in |pow(\mathcal{X})|$ and $x \in X$ then $supp(x) \subseteq supp(X)$.*
In other words, $a\#X$ and $x \in X$ does not imply $a\#x$.

Proof. It suffices to provide a counterexample. Take $\mathcal{X} = \mathbb{A}$ (the nominal set of atoms with the natural permutation action, from Subsection 2.2.1) and $X = \mathbb{A} \subseteq |\mathbb{A}|$ (the underlying set of the nominal set of all atoms, i.e. the set of all atoms!).
It is easy to check that $supp(X) = \varnothing$ and $a \in X$ and $supp(a) = \{a\} \not\subseteq \varnothing$. □

2.4.2. Strictly finitely supported powerset. Suppose $\mathcal{X}$ is a nominal set.

Definition 2.4.2. Call $X \subseteq |\mathcal{X}|$ **strictly supported** by $A \subseteq \mathbb{A}$ when

$$\forall x{\in}X.supp(x) \subseteq A.$$

If there exists some finite A which strictly supports X, then call X **strictly finitely supported**.
Write $strict(\mathcal{X})$ for the set of strictly finitely supported $X \subseteq |\mathcal{X}|$. That is:

$$strict(\mathcal{X}) = \{X \subseteq |\mathcal{X}| \mid \exists A{\subseteq}\mathbb{A}.A \text{ finite} \land X \text{ strictly supported by } A\}$$

Lemma 2.4.3. *If $X \in strict(\mathcal{X})$ then:*

(1) $\bigcup\{supp(x) \mid x{\in}X\}$ *is finite.*
(2) $\bigcup\{supp(x) \mid x{\in}X\} = supp(X)$.
(3) If $X \subseteq |\mathcal{X}|$ is strictly finitely supported then it is finitely supported.
(4) $x \in X$ implies $supp(x) \subseteq supp(X)$ (contrast this with Lemma 2.4.1).
(5) $strict(\mathcal{X})$ with the pointwise permutation action is a nominal set.

Proof. The first part is immediate since by assumption there is some finite $A{\subseteq}\mathbb{A}$ that bounds $supp(x)$ for all $x \in X$. The second part follows by an easy calculation using part 3 of Corollary 2.1.10; full details are in [Gab11a, Theorem 2.29], of which Lemma 2.4.3 is a special case. The other parts follow by definitions from the first and second parts. □

$$
\begin{array}{lrl}
(\sigma\mathbf{a}) & & a[a\mapsto u] = u \\[1em]
(\sigma\mathbf{id}) & & x[a\mapsto a] = x \\
(\sigma\#) & a\#x \Rightarrow & x[a\mapsto u] = x \\
(\sigma\alpha) & b\#x \Rightarrow & x[a\mapsto u] = ((b\ a)\cdot x)[b\mapsto u] \\
(\sigma\sigma) & a\#v \Rightarrow & x[a\mapsto u][b\mapsto v] = x[b\mapsto v][a\mapsto u[b\mapsto v]] \\[1em]
(\text{ɒ}\sigma) & a\#v \Rightarrow & p[v\leftarrow\!\!\dashv b][u\leftarrow\!\!\dashv a] = p[u[b\mapsto v]\leftarrow\!\!\dashv a][v\leftarrow\!\!\dashv b]
\end{array}
$$

Fig. 1: Nominal algebra axioms for σ and ɒ

Example 2.4.4. $\mathbb{A} \subseteq |\mathbb{A}|$ is finitely supported by $\varnothing$ but not strictly finitely supported. $\varnothing \subseteq |\mathbb{A}|$ is finitely and strictly finitely supported by $\varnothing$.

For the reader who does not like examples based on the empty set, another useful example is as follows: $\mathbb{A}\setminus\{a\}$ is finitely supported by $\{a\}$ but not strictly finitely supported. $\{a\}$ is finitely supported by $\{a\}$ and also strictly finitely supported by $\{a\}$.

Corollary 2.4.5. *If $\mathcal{X}$ is a nominal set and $\mathcal{X} \in \mathit{strict}(\mathit{strict}(\mathcal{X}))$ then $\bigcup \mathcal{X} \in \mathit{strict}(\mathcal{X})$. Or in words:* "a strictly finitely supported set of strictly finitely supported sets, is strictly finitely supported".

Proof. By routine calculations using part 4 of Lemma 2.4.3. □

2.5. The И-quantifier on nominal sets

Suppose $\mathcal{X}$ is a set with a permutation action.

Definition 2.5.1. Given a finitely supported $X \subseteq |\mathcal{X}|$ define the **new-quantifier on (nominal) sets** by

$$\text{и}a.X = \{x \mid \text{И}b.(b\ a)\cdot x \in X\}.$$

Lemma 2.5.2. *Suppose $x \in |\mathcal{X}|$. Suppose $X \subseteq |\mathcal{X}|$ is finitely supported. Then $x \in \text{и}a.X \Leftrightarrow \text{И}b.(b\ a)\cdot x \in X$.*

$\text{и}a.X$ was written $\mathsf{n}a.X$ in [GLP11, Definition 5.2], and goes back to [Gab09b] where it was written $X{-}a$. We will use $\text{и}a$ in Lemma 11.4.8 to prove things about a σ-action.

Lemma 2.5.3. $\mathit{supp}(\text{и}a.X) \subseteq \mathit{supp}(X)\setminus\{a\}$.

Proof. By a routine calculation using Corollary 2.1.10. □

Recall from Subsection 2.4.1 the notion of nominal powerset $\mathit{pow}(\mathcal{X})$.

Lemma 2.5.4. *If $X \in |\mathit{pow}(\mathcal{X})|$ then $\text{и}a.X \in |\mathit{pow}(\mathcal{X})|$.*

Proof. This amounts to showing that if $X \subseteq |\mathcal{X}|$ has finite support then so does $\text{и}a.X \subseteq |\mathcal{X}|$. This follows by Lemma 2.5.3 or direct from Theorem 2.3.1. □

I. NOMINAL DISTRIBUTIVE LATTICES WITH QUANTIFICATION

3. NOMINAL ALGEBRAS OVER NOMINAL SETS

3.1. Definition of a sigma-algebra (σ-algebra)

3.1.1. A termlike σ-algebra. Definitions 3.1.1, 3.1.4, and 3.2.1 assemble three key technical structures (see also Definitions 3.3.3 and 3.4.5).

Definition 3.1.1. A **termlike σ-algebra** is a tuple $\mathcal{X} = (|\mathcal{X}|, \cdot, \mathsf{sub}_{\mathcal{X}}, \mathsf{atm}_{\mathcal{X}})$ of:

— a nominal set $(|\mathcal{X}|, \cdot)$ which we may write just as $\mathcal{X}$; and
— an equivariant **σ-action** $\mathsf{sub}_{\mathcal{X}} : (\mathcal{X} \times \mathbb{A} \times \mathcal{X}) \to \mathcal{X}$, written $x[a\mapsto u]_{\mathcal{X}}$ or just $x[a\mapsto u]$; and
— an equivariant injection $\mathsf{atm}_{\mathcal{X}} : \mathbb{A} \to \mathcal{X}$ written $a_{\mathcal{X}}$ or just a,

such that the equalities $(\sigma\mathbf{a})$, $(\sigma\mathbf{id})$, $(\sigma\#)$, $(\sigma\alpha)$, and $(\sigma\sigma)$ of Figure 1 hold, where x, u, and v range over elements of $|\mathcal{X}|$.[12]

We may omit subscripts where $\mathcal{X}$ is understood.

Remark 3.1.2. We unpack what equivariance from Definition 2.1.6 means for the σ-action from Definition 3.1.4: for every $x \in |\mathcal{X}|$, atom a, and $u \in |\mathcal{X}|$, and for every permutation π, we have that

$$\pi\cdot(x[a\mapsto u]) = (\pi\cdot x)[\pi(a)\mapsto\pi\cdot u].$$

Similarly for the equivariant $\mathfrak{o}$-action in Definition 3.2.1 below.

Remark 3.1.3. Definition 3.1.1 is abstract. It is an axiom system. We use *nominal* algebra, because the axioms require freshness side-conditions.

Examples of termlike σ-algebras include plenty of syntax: for instance the set of terms of first-order logic with substitution; or the syntax of the untyped λ-calculus quotiented by α-equivalence with capture-avoiding substitution; or the syntax of propositional logic with quantifiers (syntax generated by $\phi ::= a \mid \bot \mid \phi \Rightarrow \phi \mid \forall a.\phi$, with capture-avoiding substitution $[a{:=}\phi]$).

However, not all termlike σ-algebras are syntax. For a huge class of extremely non-syntactic termlike σ-algebras, consider models of FM sets [Gab09b].

3.1.2. A σ-algebra

Definition 3.1.4. A **σ-algebra** is a tuple $\mathcal{X} = (|\mathcal{X}|, \cdot, \mathcal{X}^\partial, \mathsf{sub})$ of:

— A nominal set $(|\mathcal{X}|, \cdot)$ which we may write just as $\mathcal{X}$.
— A termlike σ-algebra $\mathcal{X}^\partial$.
— An equivariant **σ-action** $\mathsf{sub}_\mathcal{X} : (\mathcal{X} \times \mathbb{A} \times \mathcal{X}^\partial) \to \mathcal{X}$, written infix $x[a\mapsto u]_\mathcal{X}$ or $x[a\mapsto u]$.

such that the equalities $(\sigma\mathbf{id})$, $(\sigma\#)$, $(\sigma\alpha)$, and $(\sigma\sigma)$ of Figure 1 hold,[13] where x ranges over elements of $|\mathcal{X}|$ and u and v range over elements of $|\mathcal{X}^\partial|$.

As for termlike σ-algebras, we may omit the subscript $\mathcal{X}$. We may slightly informally say that $\mathcal{X}$ *has a σ-algebra structure over $\mathcal{X}^\partial$*.

Remark 3.1.5. Every termlike σ-algebra is a σ-algebra over itself. The canonical 'interesting' example of a σ-algebra is the syntax of predicates of first-order logic, whose substitution action is not over predicates but over the termlike σ-algebra of terms.

Not all σ-algebras are syntactic. In this paper we will see many examples of non-syntactic σ-algebras, based on the σ-powersets of Definition 3.4.5.

3.2. Definition of an amgis-algebra ($\mathfrak{o}$-algebra)

Definition 3.2.1. An **$\mathfrak{o}$-algebra** (spoken: **amgis**-algebra) is a tuple $\mathcal{P} = (|\mathcal{P}|, \cdot, \mathcal{P}^\partial, \mathsf{amgis}_\mathcal{P})$ of:

— A set with a permutation action $(|\mathcal{P}|, \cdot)$ which we may write just as $\mathcal{P}$.
— A termlike σ-algebra $\mathcal{P}^\partial$.
— An equivariant **amgis**-action $\mathsf{amgis}_\mathcal{P} : (\mathcal{P} \times \mathcal{U} \times \mathbb{A}) \to \mathcal{P}$, written infix $p[u\mapsfrom a]_\mathcal{P}$ or $p[u\mapsfrom a]$.

such that the equality $(\mathfrak{o}\sigma)$ of Figure 1 holds, where p ranges over elements of $|\mathcal{P}|$ and u and v range over elements of $|\mathcal{P}^\partial|$. We may omit the subscript $\mathcal{P}$.

Remark 3.2.2. $[u\mapsfrom a]$ looks like $[a\mapsto u]$ written backwards, and a casual glance at $(\mathfrak{o}\sigma)$ suggests that it is just $(\sigma\sigma)$ written backwards. This is not quite true: we have $u[b\mapsto v]$ on the right in $(\mathfrak{o}\sigma)$ and not '$u[v\mapsfrom b]$' (which would make no sense, since $\mathcal{P}^\partial$ has no amgis-action).

Discussion of the origin of the axioms of $\mathfrak{o}$-algebras is in Subsections 3.3 and 3.4; see also Proposition 3.3.4 and Subsection 3.4.3.

[12] Axiom $(\sigma\mathbf{id})$ might be more pedantically written as $x[a\mapsto a_\mathcal{X}] = x$.

[13] That is, the σ axioms except $(\sigma\mathbf{a})$, since we do not assume a function $\mathsf{atm}_\mathcal{X}$. Axiom $(\sigma\mathbf{id})$ can be more pedantically written as $x[a\mapsto a_{\mathcal{X}^\partial}] = x$.

In this paper, the main classes of ɒ-algebras will be those based on the ɒ-powerset of Definition 3.3.3, those based on prime filters in Definition 6.2.2, and those based on $points_{\Pi}$ in Proposition 6.2.3.

We conclude with three technical lemmas which will be useful later:

LEMMA 3.2.3. *If $\mathcal{X}$ is a σ-algebra and $b\#x$ then $x[a\mapsto b] = (b\ a)\cdot x$.*

Proof. By $(\sigma\alpha)$ $x[a\mapsto b] = ((b\ a)\cdot x)[b\mapsto b]$. We use $(\sigma\mathbf{id})$. ☐

REMARK 3.2.4. In the two papers that introduced nominal algebra [GM06a; GM06b], Lemma 3.2.3 was taken as an axiom (it was called $(\mathbf{ren}\mapsto)$) and $(\sigma\mathbf{id})$ was the lemma. In the presence of the other axioms of substitution, the two are equivalent.

LEMMA 3.2.5. *If $a\#u$ then $a\#x[a\mapsto u]$.*

Proof. Choose fresh b (so $b\#x, u$). By $(\sigma\alpha)$ $x[a\mapsto u] = ((b\ a)\cdot x)[b\mapsto u]$. Also by part 1 of Corollary 2.1.10 $(b\ a)\cdot u = u$ and by Theorem 2.3.1 $(b\ a)\cdot(x[a\mapsto u]) = ((b\ a)\cdot x)[b\mapsto (b\ a)\cdot u]$. We put this all together and we deduce that $(b\ a)\cdot(x[a\mapsto u]) = x[a\mapsto u]$. It follows by part 3 of Corollary 2.1.10 that $a \notin supp(x[a\mapsto u])$. ☐

LEMMA 3.2.6. *If $c\#x$ then $x[a\mapsto c][b\mapsto a][c\mapsto b] = (b\ a)\cdot x$.*

Proof. We use Lemma 3.2.3, Proposition 2.3.3, and Corollary 2.1.10 to reason as follows:

$$x[a\mapsto c][b\mapsto a][c\mapsto b] = ((c\ a)\cdot x)[b\mapsto a][c\mapsto b] = (c\ b)\cdot(b\ a)\cdot((c\ a)\cdot x) = (b\ a)\cdot x$$

☐

In Subsection 3.3 we explore how to move from a σ-algebra to an ɒ-algebra using nominal powersets. In Subsection 3.4 we explore how to move from an ɒ-algebra to a σ-algebra, again using nominal powersets.

3.3. Duality I: σ to ɒ

Given a σ-algebra we generate an ɒ-algebra out of its subsets. This is Proposition 3.3.4.

DEFINITION 3.3.1. Suppose $\mathcal{X} = (|\mathcal{X}|, \cdot, \mathcal{X}^{\partial}, \mathsf{sub}_{\mathcal{X}})$ is a σ-algebra.

Give subsets $p \subseteq |\mathcal{X}|$ **pointwise** actions as follows:

$$\begin{array}{rcll} \pi\cdot p &=& \{\pi\cdot x \mid x \in p\} & \\ p[u\mapsfrom a] &=& \{x \mid x[a\mapsto u] \in p\} & \qquad u \in |\mathcal{X}^{\partial}| \end{array}$$

PROPOSITION 3.3.2. *Suppose $\mathcal{X}$ is a σ-algebra and $p \subseteq |\mathcal{X}|$ and $u \in |\mathcal{X}^{\partial}|$. Then:*

— *$x \in \pi\cdot p$ if and only if $\pi^{-1}\cdot x \in p$.*
— *$x \in p[u\mapsfrom a]$ if and only if $x[a\mapsto u] \in p$.*

Proof. By easy calculations on the pointwise actions in Definition 3.3.1. ☐

DEFINITION 3.3.3. Suppose $\mathcal{X}$ is a σ-algebra. Define the **ɒ-powerset** algebra $pow_{ɒ}(\mathcal{X})$ by setting:

— $|pow_{ɒ}(\mathcal{X})|$ is the set of subsets $p \subseteq |\mathcal{X}|$ (Definition 2.4.2) with permutation action $\pi\cdot p$ following Definition 3.3.1.[14]
— $(pow_{ɒ}(\mathcal{X}))^{\partial} = \mathcal{X}^{\partial}$.
— The amgis-action $p[u\mapsfrom a]$ follows Definition 3.3.1.

PROPOSITION 3.3.4. *If $\mathcal{X}$ is a σ-algebra then $pow_{ɒ}(\mathcal{X})$ from Definition 3.3.3 is an ɒ-algebra.*

Proof. By Theorem 2.3.1 the operations are equivariant. We verify rule (ɒσ) from Figure 1:

[14]...not just the finitely-supported ones; $p \subseteq |\mathcal{X}|$ here might not have finite support and we build examples of this later in Theorem 6.1.13.

— *Property* ($\mho\sigma$). Suppose $a\#v$. Then:

$$\begin{aligned} x \in p[v{\leftarrow\!\shortmid}b][u{\leftarrow\!\shortmid}a] &\Leftrightarrow x[a{\mapsto}u][b{\mapsto}v] \in p && \text{Proposition 3.3.2} \\ &\Leftrightarrow x[b{\mapsto}v][a{\mapsto}u[b{\mapsto}v]] \in p && (\sigma\sigma),\ a\#v \\ &\Leftrightarrow x \in p[u[b{\mapsto}v]{\leftarrow\!\shortmid}a][v{\leftarrow\!\shortmid}b] && \text{Proposition 3.3.2} \end{aligned}$$

□

3.4. Duality II: $\mho$ to σ

3.4.1. The pointwise σ-action on subsets of an $\mho$-algebra

DEFINITION 3.4.1. Suppose $\mathcal{P} = (|\mathcal{P}|, \cdot, \mathcal{P}^\partial, \mathsf{amgis}_\mathcal{P})$ is an $\mho$-algebra. Give subsets $X \subseteq |\mathcal{P}|$ **pointwise** actions as follows:

$$\boxed{\begin{aligned} \pi\cdot X &= \{\pi\cdot p \mid p \in X\} \\ X[a{\mapsto}u] &= \{p \mid \text{И}c.p[u{\leftarrow\!\shortmid}c] \in (c\ a)\cdot X\} \qquad u \in |\mathcal{P}^\partial| \end{aligned}}$$

PROPOSITION 3.4.2. *Suppose $\mathcal{P}$ is an $\mho$-algebra and $X \subseteq |\mathcal{P}|$. Suppose $p \in |\mathcal{P}|$ and $u \in |\mathcal{P}^\partial|$ and $a\#u$. Then:*

(1) $p \in X[a{\mapsto}u]$ *if and only if* $\text{И}c.p[u{\leftarrow\!\shortmid}c] \in (c\ a)\cdot X$.
(2) *If furthermore $p \in |\mathcal{P}|$ has finite support*[15] *and $a\#p$, then we can simplify part 1 of this result to $p \in X[a{\mapsto}u]$ if and only if $p[u{\leftarrow\!\shortmid}a] \in X$.*
(3) $p \in \pi\cdot X$ *if and only if* $\pi^{-1}\cdot p \in X$.

Proof. (1) Direct from Definition 3.4.1.
(2) Suppose $a\#u, p$. From part 1 of this result $p \in X[a{\mapsto}u]$ if and only if $\text{И}c.p[u{\leftarrow\!\shortmid}c] \in (c\ a)\cdot X$. By Corollary 2.1.10 $(c\ a)\cdot u = u$ and $(c\ a)\cdot p = p$, so (applying $(c\ a)$ to both sides of the equality) this is if and only if $\text{И}c.p[u{\leftarrow\!\shortmid}a] \in X$, which means that $p[u{\leftarrow\!\shortmid}a] \in X$.
(3) Direct from Theorem 2.3.1.

□

LEMMA 3.4.3 (α-equivalence). *Suppose $\mathcal{P}$ is an $\mho$-algebra and $X \subseteq |\mathcal{P}|$ has finite support. If $b\#X$ then $X[a{\mapsto}u] = ((b\ a)\cdot X)[b{\mapsto}u]$.*

Proof. By part 1 of Proposition 3.4.2 $p \in X[a{\mapsto}u]$ if and only $\text{И}c.p[u{\leftarrow\!\shortmid}c] \in (c\ a)\cdot X$, and $p \in ((b\ a)\cdot X)[b{\mapsto}u]$ if and only if $\text{И}c.p[u{\leftarrow\!\shortmid}c] \in (c\ b)\cdot((b\ a)\cdot X)$. By Corollary 2.1.10 $(c\ a)\cdot X = (c\ b)\cdot((b\ a)\cdot X)$ since $b\#X$. The result follows. □

Proposition 3.4.4 is useful, amongst other things, in Lemma 3.4.8. On syntax it is known as the *substitution lemma*, but here it is about an action on sets X, and the proof is different:

PROPOSITION 3.4.4. *Suppose $\mathcal{P}$ is an $\mho$-algebra and $X \subseteq |\mathcal{P}|$ has finite support. Suppose $u, v \in |\mathcal{P}^\partial|$. Then*

$$a\#v \quad \textit{implies} \quad X[a{\mapsto}u][b{\mapsto}v] = X[b{\mapsto}v][a{\mapsto}u[b{\mapsto}v]].$$

Proof. We reason as follows, where we write $\pi = (a'\ a) \circ (b'\ b)$:

$$\begin{aligned} p \in X[a{\mapsto}u][b{\mapsto}v] &\Leftrightarrow \text{И}a', b'.p[v{\leftarrow\!\shortmid}b'][(b'\ b)\cdot u{\leftarrow\!\shortmid}a'] \in \pi\cdot X && \text{Proposition 3.4.2} \\ &\Leftrightarrow \text{И}a', b'.p[((b'\ b)\cdot u)[b'{\mapsto}v]{\leftarrow\!\shortmid}a'][v{\leftarrow\!\shortmid}b'] \in \pi\cdot X && (\mho\sigma) \\ &\Leftrightarrow \text{И}a', b'.p[u[b{\mapsto}v]{\leftarrow\!\shortmid}a'][v{\leftarrow\!\shortmid}b'] \in \pi\cdot X && (\sigma\alpha) \\ &\Leftrightarrow p \in X[b{\mapsto}v][a{\mapsto}u[b{\mapsto}v]] && \text{Proposition 3.4.2} \end{aligned}$$

□

---

[15] Our notion of $\mho$-algebra permits the possibility of p without finite support; see Definition 3.2.1. The $\mho$-algebra underlying $F(\mathcal{D})$ in Definition 8.1.5 need not have finite support; the action happens in Theorem 6.1.13 where we use Zorn's Lemma to make infinitely many choices. In contrast, the $\mho$-algebra underlying $points_\Pi$ in Definition 11.1.3 does have finite support.

3.4.2. The σ-powerset $pow_\sigma(\mathcal{P})$. Recall from Subsection 2.4.1 the *finitely supported powerset* $pow(\mathcal{X})$ of a nominal set $\mathcal{X}$.

DEFINITION 3.4.5. Suppose $\mathcal{P}$ is an $\mho$-algebra. Define the **σ-powerset** algebra $pow_\sigma(\mathcal{P})$ by setting:

— $|pow_\sigma(\mathcal{P})|$ is those $X \in |pow(\mathcal{P})|$ with the actions $\pi{\cdot}X$ and $X[a{\mapsto}u]$ from Definition 3.4.1, satisfying conditions 1 and 2 below.
— $pow_\sigma(\mathcal{P})^\partial = \mathcal{P}^\partial$.

The $X \in |pow(\mathcal{P})|$ above are restricted with conditions as follows, where $u \in |\mathcal{P}^\partial|$ and $p \in |\mathcal{P}|$:

(1) $\forall u.\text{И}a.\forall p.(p[u{\leftarrow\!\shortmid}a] \in X \Leftrightarrow p \in X)$.
(2) $\text{И}a,b.\forall p.(p[b{\leftarrow\!\shortmid}a] \in X \Leftrightarrow (b\ a){\cdot}p \in X)$.

Lemma 3.4.6 rephrases conditions 1 and 2 of Definition 3.4.5, in a simpler language, albeit one which requires the σ-action on subsets of an $\mho$-algebra from Definition 3.4.1:

LEMMA 3.4.6. *Continuing the notation of Definition 3.4.5, if $X \in |pow_\sigma(\mathcal{P})|$ then*

(1) If $a\#X$ then $X[a{\mapsto}u] = X$.
(2) If $b\#X$ then $X[a{\mapsto}b] = (b\ a){\cdot}X$.

Proof. (1) Suppose $a\#X$. By part 1 of Lemma 3.4.2 $p \in X[a{\mapsto}u]$ if and only if $\text{И}c.p[u{\leftarrow\!\shortmid}c] \in (c\ a){\cdot}X$. By Corollary 2.1.10 $(c\ a){\cdot}X = X$ and by condition 1 of Definition 3.4.5 $p[u{\leftarrow\!\shortmid}c] \in X$ if and only if $p \in X$, so this is if and only if $\text{И}c.(p \in X)$, that is $p \in X$.
(2) We combine Proposition 3.4.2 with condition 2 of Definition 3.4.5, since $a\#b$.

□

COROLLARY 3.4.7. *Suppose $X \in |pow_\sigma(\mathcal{P})|$. Then $X[a{\mapsto}a_{\mathcal{P}^\partial}] = X$.*

Proof. Suppose $b\#X$. By Lemma 3.4.3 $X[a{\mapsto}a] = ((b\ a){\cdot}X)[b{\mapsto}a]$. Note that by Proposition 2.3.3 $a\#(b\ a){\cdot}X$. By part 2 of Lemma 3.4.6 $((b\ a){\cdot}X)[b{\mapsto}a] = (b\ a){\cdot}((b\ a){\cdot}X) = X$. □

LEMMA 3.4.8. *If $X \in |pow_\sigma(\mathcal{P})|$ and $u \in |\mathcal{P}^\partial|$ then also $X[a{\mapsto}u] \in |pow_\sigma(\mathcal{P})|$.*
As a corollary, in Definition 3.4.5, $|pow_\sigma(\mathcal{P})|$ is closed under the σ-action from Definition 3.4.1.

Proof. By construction $X[a{\mapsto}u] \subseteq |\mathcal{P}|$, so we now check the properties listed in Definition 3.4.5.
By assumption in Definition 3.4.5, X is finitely supported. Finite support of $X[a{\mapsto}u]$ is from Theorem 2.3.1.
We check the conditions of Definition 3.4.5 for $X[a{\mapsto}u]$:

(1) *For fresh b (so $b\#u, X$), $X[a{\mapsto}u][b{\mapsto}v] = X[a{\mapsto}u]$.*
We use Lemma 3.4.3 to assume without loss of generality that $a\#u$. It suffices to reason as follows:
$$\begin{aligned} X[a{\mapsto}u][b{\mapsto}v] &= X[b{\mapsto}v][a{\mapsto}u[b{\mapsto}v]] && \text{Proposition 3.4.4},\ a\#v \\ &= X[b{\mapsto}v][a{\mapsto}u] && (\sigma\#),\ b\#u \\ &= X[a{\mapsto}u] && \text{Part 1 of Lemma 3.4.6},\ b\#X \end{aligned}$$
(2) *For fresh b' (so $b'\#u, v, X$) $X[a{\mapsto}u][b{\mapsto}b'] = (b'\ b){\cdot}(X[a{\mapsto}u])$.*
It suffices to reason as follows:
$$\begin{aligned} X[a{\mapsto}u][b{\mapsto}b'] &= X[b{\mapsto}b'][a{\mapsto}u[b{\mapsto}b']] && \text{Proposition 3.4.4},\ a\#b' \\ &= ((b'\ b){\cdot}X)[a{\mapsto}(b'\ b){\cdot}u] && \text{Lemma 3.4.6},\ b'\#u, X \\ &= (b'\ b){\cdot}(X[a{\mapsto}u]) && \text{Part 2 of Theorem 2.3.1} \end{aligned}$$

□

PROPOSITION 3.4.9. *If $\mathcal{P}$ is an $\mho$-algebra then $pow_\sigma(\mathcal{P})$ (Definition 3.4.5) is indeed a σ-algebra.*

Proof. By Lemma 3.4.8 the σ-action does indeed map to $|pow_\sigma(\mathcal{P})|$. By Theorem 2.3.1 so does the permutation action. It remains to check validity of the axioms from Definition 3.1.4.

— Axiom $(\sigma\mathbf{id})$ is Corollary 3.4.7.

— Axiom $(\sigma\#)$ is part 1 of Lemma 3.4.6.
— Axiom $(\sigma\alpha)$ is Lemma 3.4.3.
— Axiom $(\sigma\sigma)$ is Proposition 3.4.4.

$\square$

3.4.3. Some further remarks. The definition of $X[a\mapsto u]$ in Definition 3.4.1 is designed to make α-equivalence (Lemma 3.4.3) hold—that is, $(\sigma\alpha)$ from Figure 1.

If $\mathcal{P}$ is finitely supported so that every $p \in |\mathcal{P}|$ has finite support, then the definition can be simplified as described in part 2 of Proposition 3.4.2.

This simpler version of the definition appeared previously in [Gab11b; Gab12]. We arrived at Definition 3.4.1 as a modification and generalisation of the first definition to the case where we cannot assume that points have finite support (because of a later use of Zorn's Lemma in Theorem 6.1.13).

Interestingly, only $(\sigma\sigma)$ comes directly from the structure of the underlying $\mho$-algebra (from $(\mho\sigma)$). Other axioms are forced—$(\sigma\alpha)$ from the definition (Lemma 3.4.3), and $(\sigma\#)$ and $(\sigma\mathrm{id})$ from conditions 1 and 2 in Definition 3.4.5.

In fact, we could start from a structure satisfying just $(\sigma\sigma)$ to obtain an $\mho$-algebra using the construction in Definition 3.3.1, then move to a σ-algebra using Definition 3.4.1.

In this paper, we never need to do this; we will always start from a full σ-algebra.

4. NOMINAL POSETS

4.1. Nominal posets and fresh-finite limits

DEFINITION 4.1.1. A **nominal poset** is a tuple $\mathcal{L} = (|\mathcal{L}|, \cdot, \leq)$ where
$(|\mathcal{L}|, \cdot)$ is a nominal set, and
— $\leq \subseteq |\mathcal{L}| \times |\mathcal{L}|$ is an equivariant partial order.[16]

DEFINITION 4.1.2. Say a nominal poset $\mathcal{L}$ is **finitely fresh-complete** or has **fresh-finite limits** when:

— $\mathcal{L}$ has a top element $\top$.
— $\mathcal{L}$ has conjunctions $x\wedge y$ (a greatest lower bound for x and y).
— $\mathcal{L}$ has a-fresh limits $\bigwedge^{\#a}x$, where $\bigwedge^{\#a}x$ is greatest amongst elements x' such that $x' \leq x$ and $supp(x') \subseteq supp(x)\setminus\{a\}$.

Say $\mathcal{L}$ is **finitely cocomplete**[17] or say it has **finite colimits** when:

— $\mathcal{L}$ has a bottom element $\bot$.
— $\mathcal{L}$ has disjunctions $x\vee y$ (a least upper bound for x and y).

Lemmas 4.1.3, 4.1.4, and 4.1.5 will be useful later:

LEMMA 4.1.3. *If $b\#x$ then $\bigwedge^{\#a}x = \bigwedge^{\#b}(b\ a)\cdot x$.*

Proof. By assumption $a\#\bigwedge^{\#a}x$ and by Theorem 2.3.1 also $b\#\bigwedge^{\#a}x$, so by part 1 of Corollary 2.1.10 $\bigwedge^{\#a}x = (b\ a)\cdot\bigwedge^{\#a}x$. By part 2 of Theorem 2.3.1 $(b\ a)\cdot\bigwedge^{\#a}x = \bigwedge^{\#b}(b\ a)\cdot x$. $\square$

LEMMA 4.1.4. *$\top$, $\bot$, $x\wedge y$, $x\vee y$, and $\bigwedge^{\#a}x$ are unique if they exist.*

Proof. Since for a partial order, $x \leq y$ and $y \leq x$ imply $x = y$. $\square$

LEMMA 4.1.5. $\forall a.(x_1\wedge\ldots\wedge x_n) = (\forall a.x_1)\wedge\ldots\wedge(\forall a.x_n)$.

Proof. Both the left-hand and right-hand sides specify a greatest element z such that $a\#z$ and $z \leq x_i$ for $1{\leq}i{\leq}n$. $\square$

LEMMA 4.1.6. $\forall a.\forall b.x = \forall b.\forall a.x$.

[16] So $\leq$ is transitive, reflexive, and antisymmetric, and $x \leq y$ if and only if $\pi\cdot x \leq \pi\cdot y$.
[17] There is also a notion of finitely fresh-cocomplete, but we will not need it.

Proof. Both the left-hand and right-hand sides specify a greatest element z such that $a\#z$ and $b\#z$ and $z \leq x$. □

LEMMA 4.1.7. *Suppose $\mathcal{L}$ is a finitely fresh-complete nominal poset with a monotone σ-action. Then $x \leq x'$ implies $\forall a.x \leq \forall a.x'$.*

Proof. Suppose $x \leq x'$. By assumption $\forall a.x \leq x$ and $a\#\forall a.x$. But then also $\forall a.x \leq x'$ and $a\#\forall a.x$, so $\forall a.x \leq \forall a.x'$. □

NOTATION 4.1.8. We may write $\forall a_1, \ldots, a_n.x$ for $\forall a_1. \ldots .\forall a_n.x$.

4.2. More characterisations of fresh-finite limits

$\bigwedge^{\#a}x$ from Definition 4.1.2 is greatest in the set $\{x' \mid x' \leq x \wedge a\#x'\}$. This is finitely supported but not *strictly* finitely supported. Definition 4.2.1 and Proposition 4.2.3 will characterise $\bigwedge^{\#a}x$ further as a limit of a strictly finitely-supported set.

This turns out to be important, and we use it in part 2 of Lemma 5.1.1, whose use in Lemma 5.2.3 is important for proving Proposition 5.2.6 (closure of pow_σ under quantification).

Definition 4.2.5 and Proposition 4.2.7 then characterise $\bigwedge^{\#a}x$ in two ways: as a limit of a permutation orbit, and using the И-quantifier. This ties $\bigwedge^{\#a}x$ to Proposition 5.2.8 and Remark 5.2.9 and to the opening discussion of Subsection 6.1. See also [Gab09b; GC11] and computational studies such as [BBKL12], with their emphasis on studying nominal sets in terms of their permutation orbits.

DEFINITION 4.2.1. Suppose $\mathcal{L}$ is a nominal poset and $x \in |\mathcal{L}|$. Consider the set

$$B = \{x' \in |\mathcal{L}| \mid supp(x') \subseteq S \ \wedge \ x' \leq x\}.$$

Then write $\bigwedge^{\subseteq S}x$ for the $\leq$-greatest element of B, if this exists. We call this the **S-strict limit** of x.

REMARK 4.2.2. So:

— $\bigwedge^{\#a}x$ is the greatest x' beneath x such that $a \notin supp(x')$.
— $\bigwedge^{\subseteq supp(x)\setminus\{a\}}x$ is greatest x' beneath x such that $supp(x') \subseteq supp(x)\setminus\{a\}$.

It is not *a priori* evident that these two notions must coincide. However, they often do, as we will now show.

When they coincide, $\bigwedge^{\subseteq S}$ can be easier to work with than $\bigwedge^{\#a}$ because it is the limit of a strictly finitely supported set, which have properties that finitely supported sets do not; see Lemma 2.4.3.

PROPOSITION 4.2.3. *If $\bigwedge^{\#a}x$ exists then so does $\bigwedge^{\subseteq supp(x)\setminus\{a\}}x$ and they are equal.*

Proof. Suppose $\bigwedge^{\#a}x$ exists. By construction $supp(\bigwedge^{\#a}x) \subseteq supp(x)\setminus\{a\}$ and $\bigwedge^{\#a}x \leq x$. Therefore $\bigwedge^{\#a}x \in B$ (notation from Definition 4.2.1). Also by construction $x' \leq \bigwedge^{\#a}x$ for every $x' \in B$, since if $supp(x') \subseteq supp(x)\setminus\{a\}$ then certainly $a\#x'$. It follows that $\bigwedge^{\#a}x$ is greatest in B. □

REMARK 4.2.4. It is easy to prove that if $x \leq y$ then $\bigwedge^{\subseteq S}x \leq \bigwedge^{\subseteq S}y$ (so $\bigwedge^{\subseteq S}$ is monotone or *functorial*). Another plausible definition for $\bigwedge^{\subseteq a}$ is that

$$\bigwedge\nolimits^{\subseteq a}x \quad \text{be the } \leq\text{-greatest element of} \quad \{x' \mid supp(x') \subseteq supp(x)\setminus\{a\} \wedge x' \leq x\}. \tag{1}$$

However, monotonicity is then not so obvious, though in view of Proposition 4.2.3 monotonicity does hold of (1), for the cases we care about.

Recall from Notation 2.1.3 the definition of *fix*.

DEFINITION 4.2.5. Following [Gab09b; GC11] define $x\circlearrowleft^a$ by

$$\begin{aligned} x\circlearrowleft^a &= \{\pi\cdot x \mid \pi \in \mathit{fix}(supp(x)\setminus\{a\})\}. \\ &= \{x\} \cup \{(b\ a)\cdot x \mid b\#x\} \end{aligned}$$

Then consider the $\leq$-greatest lower bound of $x\circlearrowleft^a$, if this exists.

REMARK 4.2.6. So we can rewrite this as follows:

— $\bigwedge^{\#a} x$ is the greatest x' beneath x such that $a \notin \mathit{supp}(x')$.
— Definition 4.2.5 specifies the greatest x' beneath x and beneath $(b\ a)\cdot x$ for every $b\#x$.

Proposition 4.2.7 is needed for Proposition 5.2.8:

PROPOSITION 4.2.7. *If x is an element of a nominal poset $\mathcal{L}$ and a greatest lower bound for $x\circlearrowleft^{a}$ exists in $\mathcal{L}$, then so does $\bigwedge^{\#a} x$ and they are equal, and vice versa.*
As a corollary, if $\mathcal{L}$ has fresh-finite limits then $\bigwedge^{\#a} x$ is greatest amongst z such that $\reflectbox{N}b.z \leq (b\ a)\cdot x$.

Proof. If $\bigwedge^{\#a} x$ exists then $\bigwedge^{\#a} x \leq x$ and $a\#\bigwedge^{\#a} x$. It follows by equivariance and Corollary 2.1.10 that $\bigwedge^{\#a} x \leq \pi\cdot x$ for every $\pi \in \mathit{fix}(\mathit{supp}(x)\setminus\{a\}$.
Conversely suppose z is greatest such that $\forall \pi{\in}\mathit{fix}(\mathit{supp}(x)\setminus\{a\}).z \leq \pi\cdot x$. Then z is unique and by Theorem 2.3.1 $\mathit{supp}(z) \subseteq \mathit{supp}(x)\setminus\{a\}$.
Now suppose $\mathcal{L}$ has fresh-finite limits. Then $\bigwedge^{\#a} x \leq x$ and $a\#\bigwedge^{\#a} x$ so that by Corollary 2.1.10 $\reflectbox{N}b.\big(\bigwedge^{\#a} x = (b\ a)\cdot\bigwedge^{\#a} x \leq (b\ a)\cdot x\big)$. Also, if $\reflectbox{N}b.z \leq (b\ a)\cdot x$ then z is a lower bound for $x\circlearrowleft^{a}$ so that by the first part of this result, $z \leq \bigwedge^{\#a} x$. $\square$

REMARK 4.2.8. So we have seen three natural notions of 'fresh-finite limit' in nominal posets:

— Fresh-finite limits from Definition 4.1.2.
— *Strict* fresh-finite limits from Definition 4.2.1.
— Limits of permutation orbits.

By Proposition 4.2.7 the first and the third are identical, and by Proposition 4.2.3 the second one exists and is equal to the first (and the third) where the first exists. For a converse see Appendix A.2.

4.3. Compatible σ-structure

DEFINITION 4.3.1. Say that a finitely fresh-complete and finitely cocomplete nominal poset $\mathcal{L} = (|\mathcal{L}|, \cdot, \leq)$ has a **compatible** σ-algebra structure when it is also a σ-algebra $(|\mathcal{L}|, \cdot, \mathcal{L}^{\partial}, \mathsf{sub}_{\mathcal{L}})$ and in addition

$$\boxed{\begin{aligned}(x\wedge y)[a\mapsto u] &= (x[a\mapsto u])\wedge(y[a\mapsto u])\\ (x\vee y)[a\mapsto u] &= (x[a\mapsto u])\vee(y[a\mapsto u])\\ b\#u \Rightarrow (\textstyle\bigwedge^{\#b} x)[a\mapsto u] &= \textstyle\bigwedge^{\#b}(x[a\mapsto u])\end{aligned}}$$

where $x, y \in |\mathcal{L}|$ and $u \in |\mathcal{L}^{\partial}|$, where $\wedge$, $\vee$, and $\forall b$ exist.
Call the σ-action **monotone** when

$$\boxed{x \leq y \quad \text{implies} \quad x[a\mapsto u] \leq y[a\mapsto u].}$$

LEMMA 4.3.2. *Continuing Definition 4.3.1, if the σ-structure is compatible then it is monotone.*

Proof. It is a fact that $x \leq y$ if and only if $x \wedge y = x$. The result follows. $\square$

LEMMA 4.3.3. *Suppose $\mathcal{L}$ is a nominal poset with a monotone σ-action, and suppose $a\#z$ for $z \in |\mathcal{L}|$.*
Then if $z \leq x$ then z is a lower bound for $\{x[a\mapsto u] \mid u \in |\mathcal{L}^{\partial}|\}$, and as a particular corollary,

$$\forall a.x \leq x[a\mapsto u]$$

for every $x \in |\mathcal{L}|$ and $u \in |\mathcal{L}^{\partial}|$.

Proof. By monotonicity $z[a\mapsto u] \leq x[a\mapsto u]$ for every $u \in |\mathcal{L}^{\partial}|$. By $(\sigma\#)$ also $z = z[a\mapsto u]$. The corollary follows just noting that by the definition of fresh-finite limit in Definition 4.1.2, $\forall a.x \leq x$ and $a\#\forall a.x$. $\square$

LEMMA 4.3.4. $a\#\{x[a\mapsto u] \mid u \in |\mathcal{L}^{\partial}|\}$

Proof. We use part 3 of Corollary 2.1.10. Choose fresh b (so $b\#x$). Then we reason as follows:

$$\begin{array}{rll}(b\ a)\cdot\{x[a\mapsto u] \mid u\in|\mathcal{L}^\partial|\} &= \{(b\ a)\cdot(x[a\mapsto u]) \mid u\in|\mathcal{L}^\partial|\} & \text{Pointwise action}\\ &= \{((b\ a)\cdot x)[b\mapsto(b\ a)\cdot u] \mid u\in|\mathcal{L}^\partial|\} & \text{Theorem 2.3.1}\\ &= \{((b\ a)\cdot x)[b\mapsto u] \mid u\in|\mathcal{L}^\partial|\} & \pi\cdot|\mathcal{L}^\partial| = |\mathcal{L}^\partial|\\ &= \{x[a\mapsto u] \mid u\in|\mathcal{L}^\partial|\} & (\sigma\alpha),\ b\#x\end{array}$$

□

Intuitively, universal quantification is an infinite intersection. Proposition 4.3.5 makes that formal for nominal posets:

PROPOSITION 4.3.5. *Suppose $\mathcal{L}$ is a nominal poset with a monotone σ-action (Definition 4.3.1), and suppose $x\in|\mathcal{L}|$. Then:*

(1) *If $\bigwedge^{\#a}x$ exists then so does $\bigwedge_{u\in|\mathcal{L}^\partial|}x[a\mapsto u]$ the limit for $\{x[a\mapsto u] \mid u\in|\mathcal{L}^\partial|\}$, and they are equal. In symbols:*

$$\bigwedge\nolimits^{\#a}x = \bigwedge_{u\in|\mathcal{L}^\partial|} x[a\mapsto u].$$

(2) *If $\bigwedge_u x[a\mapsto u]$ exists then so does $\bigwedge^{\#a}x$, and they are equal.*

Proof. By Lemma 4.3.3 $\bigwedge^{\#a}x$ is a lower bound for $\{x[a\mapsto u] \mid u\in|\mathcal{L}^\partial|\}$.

Now suppose z is any other lower bound, that is: $z\leq x[a\mapsto u]$ for every $u\in|\mathcal{L}^\partial|$. Note that we do not know *a priori* that $a\#z$.

Choose b fresh (so $b\#z,x$) and take $u=b$. Then $z\leq x[a\mapsto b] \stackrel{\text{Lem 3.2.3}}{=} (b\ a)\cdot x$. Since $b\#z$ it follows that $z\leq\bigwedge^{\#b}(b\ a)\cdot x \stackrel{\text{Lem. 4.1.3}}{=} \bigwedge^{\#a}x$. So $\bigwedge^{\#a}x=\bigwedge_u x[a\mapsto u]$.

Now suppose that $\bigwedge_u x[a\mapsto u]$ exists. By Lemma 4.3.4 and part 2 of Theorem 2.3.1 we have that $a\#\bigwedge_u x[a\mapsto u]$. Also by assumption $\bigwedge_u x[a\mapsto u]\leq x[a\mapsto a] \stackrel{(\sigma\mathbf{id})}{=} x$. Thus $\bigwedge_u x[a\mapsto u]$ is an $a\#$lower bound for x.

Now suppose $z\leq x$ and $a\#z$; we need to show that $z\leq\bigwedge_u x[a\mapsto u]$. This is direct from Lemma 4.3.3. □

We also mention a characterisation of $\bigwedge^{\#a}x$ using a 'smaller' conjunction which does not depend on most of $\mathfrak{U}$ (see also Remark 5.2.9):

PROPOSITION 4.3.6. *Suppose $\mathcal{L}$ is a nominal poset with a monotone σ-action and $x\in|\mathcal{L}|$. Then:*

(1) *If $\bigwedge^{\#a}x$ exists then so does $\bigwedge_n x[a\mapsto n]$ where n ranges over all atoms, and they are equal.*
(2) *If $\bigwedge_n x[a\mapsto n]$ exists then so does $\bigwedge^{\#a}x$, and they are equal.*[18]

Proof. As the proof of Proposition 4.3.5. The important point is that by Lemma 4.3.4 and part 2 of Theorem 2.3.1 we have $a\#\bigwedge_n x[a\mapsto n]$. □

4.4. Definition of a nominal distributive lattice with ∀

DEFINITION 4.4.1. Suppose $\mathcal{L}$ is a fresh-finitely complete and finitely cocomplete nominal poset. Call $\mathcal{L}$ **distributive** when

$$\begin{array}{rl} & x\vee(y\wedge z) = (x\vee y)\wedge(x\vee z)\\ a\#x\Rightarrow & x\vee\bigwedge^{\#a}y = \bigwedge^{\#a}(x\vee y)\end{array}\qquad \text{for every } x,y,z\in|\mathcal{L}|.$$

[18]Strictly speaking we should write $\bigwedge_n x[a\mapsto\mathsf{atm}_{\mathfrak{U}}(n)]$. See the notation in Definition 3.1.1.

Remark 4.4.2. Definition 4.4.1 generalises the usual notion of distributivity; $\vee$ distributes over $\wedge$ and also over $\forall a$ (subject to a typical nominal algebra freshness side-condition), which we have already seen exhibited as an infinite intersection in Proposition 4.3.5.[19]

Definition 4.4.3. A **nominal distributive lattice with** $\forall$ is a tuple $\mathcal{D} = (|\mathcal{D}|, \cdot, \leq, \mathcal{D}^\partial, \mathsf{sub}_\mathcal{D})$ such that:

- $(|\mathcal{D}|, \cdot, \leq)$ is a nominal poset.
- $\mathcal{D}$ has fresh-finite limits and finite colimits, and is distributive.
- $(|\mathcal{D}|, \cdot, \mathcal{D}^\partial, \mathsf{sub}_\mathcal{D})$ is a σ-algebra (Definition 3.1.4), and the σ-algebra structure is compatible (Definition 4.3.1).

Recall the notions of σ-algebra and termlike σ-algebra from Definitions 3.1.1 and 3.1.4:

Definition 4.4.4. Suppose $\mathcal{X}$ and $\mathcal{X}'$ are σ-algebras. Call a pair of functions $f = (f_\mathcal{X}, f^\partial_\mathcal{X})$ where $f_\mathcal{X} \in |\mathcal{X}| \to |\mathcal{X}'|$ and $f^\partial_\mathcal{X} \in |\mathcal{X}^\partial| \to |\mathcal{X}'^\partial|$ a **(σ-algebra) morphism** from $\mathcal{X}$ to $\mathcal{X}'$ when:

(1) $f^\partial_\mathcal{X}(a_{\mathcal{X}^\partial}) = a_{\mathcal{X}'^\partial}$ (so f maps atoms to atoms).
(2) $f^\partial_\mathcal{X}(\pi{\cdot}u) = \pi{\cdot}f^\partial_\mathcal{X}(u)$ and $f_\mathcal{X}(\pi{\cdot}x) = \pi{\cdot}f_\mathcal{X}(x)$ (so f is equivariant).
(3) $f^\partial_\mathcal{X}(u'[a{\mapsto}u]) = f^\partial_\mathcal{X}(u')[a{\mapsto}f^\partial_\mathcal{X}(u)]$ and $f_\mathcal{X}(x[a{\mapsto}u]) = f_\mathcal{X}(x)[a{\mapsto}f^\partial_\mathcal{X}(u)]$ (so f commutes with the σ-action).

If $\mathcal{X}$ is termlike then we insist $\mathcal{X} = \mathcal{X}^\partial$ and we insist that $f_\mathcal{X} = f^\partial_\mathcal{X}$.

We may omit the subscripts, writing for instance f and f^∂, or even $f = (f, f^\partial)$, where the meaning is clear.

Definition 4.4.5. Suppose $\mathcal{D}$ and $\mathcal{D}'$ are nominal distributive lattices with $\forall$. Call a morphism $f = (f_\mathcal{D}, f^\partial_\mathcal{D}) : \mathcal{D} \to \mathcal{D}'$ of underlying σ-algebras (Definition 4.4.4) a **morphism** of nominal distributive lattices with $\forall$ when f commutes with fresh-finite limits and with finite colimits:

(1) $f(\top) = \top$, and $f(x{\wedge}y) = f(x){\wedge}f(y)$ and $f(\forall a.x) = \forall a.f(x)$, and
(2) $f(\bot) = \bot$ and $f(x{\vee}y) = f(x){\vee}f(y)$.

Write $\mathsf{nDi}\forall$ for the category of nominal distributive lattices with $\forall$ and morphisms between them.

4.5. Impredicative nominal distributive lattices

We are interested in modelling the λ-calculus, so we care about lattices where the substitution action is over *itself*. Therefore we introduce *impredicative* nominal distributive lattices with $\forall$: this is Definition 4.5.1.

Recall the notion of termlike σ-algebra $\mathcal{U}$ from Definition 3.1.1, and the notion of a nominal distributive lattice with $\forall$ from Definition 4.4.3.

Definition 4.5.1. An **impredicative** nominal distributive lattice with $\forall$ is a tuple $(\mathcal{D}, \partial_\mathcal{D})$ where:

- $\mathcal{D} \in \mathsf{nDi}\forall$ is a nominal distributive lattice with $\forall$ (Definition 4.4.3).
- $(\partial_\mathcal{D}, id) : \mathcal{D}^\partial \to \mathcal{D}$ is a morphism of σ-algebras (Definition 4.4.4).

Remark 4.5.2. So $\mathcal{D}$ is impredicative when the things we substitute *for*—the $u \in |\mathcal{D}^\partial|$ in $x[a{\mapsto}u]$—can be transferred over to the things we substitute in—the $x \in |\mathcal{D}|$ in $x[a{\mapsto}u]$.

Given $u \in |\mathcal{D}^\partial|$, we can obtain $\partial_\mathcal{D} u \in |\mathcal{D}|$ and so write (for instance) $(\partial_\mathcal{D} u)[a{\mapsto}u]$. This is not quite λ-calculus self-application, but we are moving in that direction.

[19] A dual version of part 1 of Definition 4.4.1 is $x \wedge (y{\vee}z) = (x{\wedge}y){\vee}(x{\wedge}z)$ and by a standard argument [DP02, Lemma 4.3] the two are equivalent.

NOTATION 4.5.3. We introduce some notation for Definition 4.5.1:

— We may write $\partial_{\mathcal{D}}$ for $(\partial_{\mathcal{D}}, id)$.
— We may drop subscripts and write ∂u for $\partial_{\mathcal{D}} u$ where $u \in |\mathcal{D}^{\partial}|$.
— We may write ∂a for $\partial_{\mathcal{D}}(a_{\mathcal{D}^{\partial}})$ where $a_{\mathcal{D}^{\partial}}$ is itself shorthand for $\mathsf{atm}_{\mathcal{D}^{\partial}}(a)$ from Definition 3.1.1.[20]
— We may write $\partial\mathcal{D}$ for $\{\partial u \mid u \in |\mathcal{D}^{\partial}|\} \subseteq |\mathcal{D}|$ and call this set the **programs** of $\mathcal{D}$.

REMARK 4.5.4. It might help to break down the notation a little:

— $\mathbb{A}$ injects into $\mathcal{D}^{\partial}$ via an injection $\mathsf{atm}_{\mathcal{D}^{\partial}}$ (this is the equivariant injection specified in Definition 3.1.1).
— $\mathcal{D}^{\partial}$ maps to $\mathcal{D}$ via $\partial_{\mathcal{D}}$.
— Thus we obtain $\partial a \in \partial\mathcal{D}$—an atom-as-a-program—living in a sub-σ-algebra of $\mathcal{D}$ which is an image of $\mathcal{D}^{\partial}$, and which we call the *programs* of $\mathcal{D}$.

Lemma 4.5.5 is a routine sanity check that the definitions match up sensibly. It will be useful later:

LEMMA 4.5.5. *Suppose $\mathcal{D}$ is impredicative and $u \in |\mathcal{D}^{\partial}|$. Then $(\partial_{\mathcal{D}} a_{\mathcal{D}^{\partial}})[a{\mapsto}u] = \partial_{\mathcal{D}} u$.*

Proof. By assumption $\partial_{\mathcal{D}}$ is a morphism of σ-algebras from D^{∂} to $\mathcal{D}$. By Definition 4.4.4 $(\partial_{\mathcal{D}} a_{\mathcal{D}^{\partial}})[a{\mapsto}u] = \partial_{\mathcal{D}}(a_{\mathcal{D}^{\partial}}[a{\mapsto}u])$ (using the third condition, noting that $u = id(u)$). The result follows by $(\sigma\mathbf{a})$ from Figure 1 for $\mathcal{D}^{\partial}$. □

REMARK 4.5.6. Definition 4.5.1 can be looked at in some interesting ways:

— $\mathcal{D}$ is impredicative when it has substitution $x[a{\mapsto}u]$ over a substructure of itself.
— $\mathcal{D}$ is impredicative when its quantifier $\forall a.x$ quantifies over a sub-σ-structure of $\mathcal{D}$. Thus, $|\partial\mathcal{D}| \subseteq |\mathcal{D}|$ is the set of things we quantify over when we write $\forall a.x$, if $\mathcal{D}$ is impredicative.

REMARK 4.5.7. Note that the programs of $\mathcal{D}$ need not be closed under the logical structure like $\wedge$, $\vee$, and $\forall$.

So for instance $x, x' \in \partial\mathcal{D}$ does not imply $x \vee x' \in \partial\mathcal{D}$ and it is not necessarily the case that $\bot \in \partial\mathcal{D}$, and so on. We do not forbid this either.

DEFINITION 4.5.8. Suppose $\mathcal{D}$ and $\mathcal{D}'$ are impredicative nominal distributive lattices with $\forall$.

Call $f = (f_{\mathcal{D}}, f_{\mathcal{D}}^{\partial}) : \mathcal{D} \to \mathcal{D}'$ a **morphism** in $\mathsf{inDi}\forall$ when it is a morphism in $\mathsf{nDi}\forall$ (Definition 4.4.5) and when in addition:

(3) $f_{\mathcal{D}} \circ \partial_{\mathcal{D}} = \partial_{\mathcal{D}'} \circ f_{\mathcal{D}}^{\partial}$. That is,

$$f_{\mathcal{D}}(\partial_{\mathcal{D}} u) = \partial_{\mathcal{D}'}(f_{\mathcal{D}}^{\partial}(u)) \ \text{ for every } \ u \in |\mathcal{D}^{\partial}|.$$

Definition 4.5.9 extends Definition 4.4.5:

DEFINITION 4.5.9. Write $\mathsf{inDi}\forall$ for the category of **impredicative nominal distributive lattices with** $\forall$, and morphisms between them.
As standard write $\mathcal{D} \in \mathsf{inDi}\forall$ for "$\mathcal{D}$ is an impredicative nominal distributive lattice with $\forall$" and $f : \mathcal{D} \to \mathcal{D}' \in \mathsf{inDi}\forall$ for "$\mathcal{D}, \mathcal{D}' \in \mathsf{inDi}\forall$ and f is a morphism in $\mathsf{inDi}\forall$ from $\mathcal{D}$ to $\mathcal{D}'$".

REMARK 4.5.10. We continue the notation of Definition 4.5.9 and the discussion of Remark 4.5.4. Suppose $f : \mathcal{D} \to \mathcal{D}' \in \mathsf{inDi}\forall$ is a morphism.

Note that $f_{\mathcal{D}}(\partial_{\mathcal{D}} a_{\mathcal{D}^{\partial}}) = \partial_{\mathcal{D}'}(a_{\mathcal{D}'^{\partial}})$. In informal words we can say that f maps atoms-as-programs (Remark 4.5.4) in $\mathcal{D}$ to themselves in $\mathcal{D}'$. In symbols we can be even more brief:

$$f(\partial a) = \partial a.$$

[20] Atoms get mapped into $\mathcal{D}^{\partial}$ by $\mathsf{atm}_{\mathcal{D}^{\partial}}$, and $\mathcal{D}^{\partial}$ gets mapped into $\mathcal{D}$ by $\partial_{\mathcal{D}}$ ... so atoms get mapped into $\mathcal{D}$.

We informally trace through how this happens. By condition 1 of Definition 4.4.4 f maps an atom in $\mathcal{D}^\partial$ to it incarnation in $\mathcal{D}'^\partial$. By condition 3 of Definition 4.5.8 these are mapped to atoms-as-programs in $\mathcal{D}^\partial$ and $\mathcal{D}'^\partial$ respectively.

5. THE σ-POWERSET AS A NOMINAL DISTRIBUTIVE LATTICE WITH ∀

We saw in Proposition 3.4.9 how the nominal powerset of an $\mho$-algebra $\mathcal{P}$ generates a σ-algebra $pow_\sigma(\mathcal{P})$ (Definition 3.4.5). But powersets are also a lattice under subset inclusion, so perhaps $pow_\sigma(\mathcal{P})$ has more structure?

In fact, $pow_\sigma(\mathcal{P})$ is a nominal distributive lattice with ∀ (Definition 4.4.3). This is Theorem 5.2.10.

5.1. Basic sets operations

Suppose $\mathcal{P} = (|\mathcal{P}|, \cdot, \mathcal{P}^\partial, \mathsf{amgis}_\mathcal{P})$ is an $\mho$-algebra. Recall the definition of $pow_\sigma(\mathcal{P})$ from Definition 3.4.5.

Lemmas 5.1.1 and 5.1.2 are technical correctness properties of pow_σ. The proofs are not entirely obvious, but they follow the same patterns of pointwise calculations on sets.

This subsection covers the simpler cases; in Subsection 5.2 we move on to quantification.

LEMMA 5.1.1. *Suppose $\mathcal{X} \subseteq |pow_\sigma(\mathcal{P})|$. Then:*

(*1*) *If $\mathcal{X}$ is strictly finitely supported (Definition 2.4.2) then*

$$(\bigcap_{X\in\mathcal{X}} X)[a\mapsto u] = \bigcap_{X\in\mathcal{X}} (X[a\mapsto u]).$$

In words: σ commutes with strictly finitely supported sets intersections.

(*2*) *If $\mathcal{X}$ is strictly finitely supported then*

$$(\bigcup_{X\in\mathcal{X}} X)[a\mapsto u] = \bigcup_{X\in\mathcal{X}} (X[a\mapsto u]).$$

In words: σ commutes with strictly finitely supported sets unions.

(*3*) *For any X,*

$$\pi\cdot \bigcap_{X\in\mathcal{X}} X = \bigcap_{X\in\mathcal{X}} \pi\cdot X \quad \text{and} \quad \pi\cdot \bigcup_{X\in\mathcal{X}} X = \bigcup_{X\in\mathcal{X}} \pi\cdot X.$$

In words: intersections and unions are equivariant.

(*4*) *If $X \subseteq Y$ then $X[a\mapsto u] \subseteq Y[a\mapsto u]$. In words: σ is monotone (Definition 4.3.1).*

Proof. For part 1 we reason as follows:

$$\begin{aligned}
p \in (\textstyle\bigcap_{X\in\mathcal{X}} X)[a\mapsto u] &\Leftrightarrow \text{И}c.p[u\leftarrow c] \in \textstyle\bigcap_{X\in\mathcal{X}} (c\ a)\cdot X && \text{Prop 3.4.2, Thm 2.3.1}\\
&\Leftrightarrow \text{И}c.\forall X\in\mathcal{X}.p[u\leftarrow c] \in (c\ a)\cdot X && \text{Fact}\\
p \in \textstyle\bigcap_{X\in\mathcal{X}} (X[a\mapsto u]) &\Leftrightarrow \forall X\in\mathcal{X}.p \in X[a\mapsto u] && \text{Fact}\\
&\Leftrightarrow \forall X\in\mathcal{X}.\text{И}c.p[u\leftarrow c] \in (c\ a)\cdot X && \text{Proposition 3.4.2}
\end{aligned}$$

We note that by Lemma 2.4.3, $c\#\mathcal{X}$ if and only if $c\#X$ for every $X \in \mathcal{X}$. This allows us to swap the ∀ and the И quantifier, and the result follows.

The second and third parts are similar. Part 4 follows from part 1 as in the proof of Lemma 4.3.2. □

LEMMA 5.1.2.— *$\varnothing$ and $|\mathcal{P}|$ are in $|pow_\sigma(\mathcal{P})|$ and these are least and greatest elements in the subset inclusion ordering.*

— *If X and Y are in $|pow_\sigma(\mathcal{P})|$ then so are $X \cap Y$ and $X \cup Y$ and these are greatest lower bounds and least upper bounds in the subset inclusion ordering.*

Proof. We check the properties listed in Definition 3.4.5 for $X\cap Y$; the case of $X\cup Y$ is similar and the cases of $\varnothing$ and $|\mathcal{P}|$ are even easier. We check that $X\cap Y$ is a greatest lower bound for $\{X,Y\}$ just as for ordinary sets. $X\cap Y$ has finite support by Theorem 2.3.1.

(1) *If a is fresh (so $a\#X,Y,u$) then $(X\cap Y)[a\mapsto u]=X\cap Y$.* By Lemmas 5.1.1 and 3.4.6 $(X\cap Y)[a\mapsto u]=(X[a\mapsto u])\cap(Y[a\mapsto u])=X\cap Y$.
(2) *If b is fresh (so $b\#X,Y$) then $(X\cap Y)[a\mapsto b]=(b\ a)\cdot(X\cap Y)$.* We reason as follows:

$$\begin{aligned}(X\cap Y)[a\mapsto b]&=(X[a\mapsto b])\cap(Y[a\mapsto b]) &&\text{Lemma 5.1.1}\\ &=((b\ a)\cdot X)\cap((b\ a)\cdot Y) &&\text{Part 2 of Lemma 3.4.6}\\ &=(b\ a)\cdot(X\cap Y) &&\text{Theorem 2.3.1}\end{aligned}$$

□

5.2. Sets quantification

We now explore quantification. This is where we part company from Boolean algebras.

Suppose $\mathcal{P}=(|\mathcal{P}|,\cdot,\mathcal{P}^{\partial},\mathsf{amgis}_{\mathcal{P}})$ is an $\mho$-algebra. Recall the definitions of $pow(\mathcal{P})$ Subsection 2.4.1 and of $pow_{\sigma}(\mathcal{P})$ from Definition 3.4.5.

DEFINITION 5.2.1. If $X\in|pow(\mathcal{P})|$ then define

$$\boxed{\bigcap\nolimits^{\#a}X=\bigcup\{X'\in pow(\mathcal{P})\mid X'\subseteq X\text{ and }a\#X'\}.}$$

Definition 5.2.1 contains a wealth of structure. Clearly, this is our candidate for quantification over a in $pow_{\sigma}(\mathcal{P})$. But first, note that Definition 5.2.1 takes place in $pow(\mathcal{P})$, not $pow_{\sigma}(\mathcal{P})$. This is the set of all finitely-supported subsets of $\mathcal{P}$, not the more restricted set of subsets forming a σ-algebra from Definition 3.4.5.

This is because it is easy to see that a set is in $pow(\mathcal{P})$ (just from Theorem 2.3.1) and harder to prove that a set exists in $pow_{\sigma}(\mathcal{P})$, because for a set to be in the latter it must satisfy two extra conditions (Definition 3.4.5) which are not trivial to prove.

So the proofs work by starting in $pow(\mathcal{P})$, then using Corollary 5.2.2 to present $\bigcap^{\#a}X$ as a strictly finitely supported intersection, using *that* to prove Lemma 5.2.3,[21] and then finally we promote the set from $pow(\mathcal{P})$ to $pow_{\sigma}(\mathcal{P})$ in Proposition 5.2.6. The other main result of this subsection is Proposition 5.2.8, which gives useful alternative characterisations of Definition 5.2.1.

COROLLARY 5.2.2. *If $X\in pow(\mathcal{P})$ then*

$$\bigcap\nolimits^{\#a}X=\bigcup\{X'\in pow(\mathcal{P})\mid X'\subseteq X\textit{ and }supp(X')\subseteq supp(X)\setminus\{a\}\}.$$

Proof. From Proposition 4.2.3. □

LEMMA 5.2.3. *Suppose $X\in|pow(\mathcal{P})|$ and $v\in|\mathcal{P}^{\partial}|$ and $a\#v$. Then*

$$(\bigcap\nolimits^{\#a}X)[b\mapsto v]=\bigcap\nolimits^{\#a}(X[b\mapsto v]).$$

Proof. We combine Definition 5.2.1 with Lemma 5.1.1 and Corollary 5.2.2, since

$$\bigcup\{X'\in|pow(\mathcal{P})|\mid X'\subseteq X\text{ and }supp(X')\subseteq supp(X)\setminus\{a\}\}$$

is strictly finitely supported by $supp(X)\setminus\{a\}$. □

[21] We could take Definition 5.2.1 to be $\bigcap^{\#a}X=\bigcap_{u\in|\mathcal{P}^{\partial}|}X[a\mapsto u]$—see line 2 of Proposition 5.2.8—but this has the disadvantage that it is not necessarily an intersection of a strictly finitely supported set.

REMARK 5.2.4. Lemmas 5.2.3 and 5.1.1 for $\{X[a{\mapsto}u] \mid u{\in}|\mathcal{P}^{\partial}|\}$ are different. It might be useful to write out why:

$$\begin{array}{ll}\text{Lemma 5.1.1 delivers} & (\bigcap^{\#a}X)[b{\mapsto}v] = \bigcap\{X[a{\mapsto}u][b{\mapsto}v] \mid u{\in}|\mathcal{P}^{\partial}|\}\\ \text{Lemma 5.2.3 delivers} & (\bigcap^{\#a}X)[b{\mapsto}v] = \bigcap\{X[b{\mapsto}v][a{\mapsto}u] \mid u{\in}|\mathcal{P}^{\partial}|\}\end{array}$$

LEMMA 5.2.5. *Suppose* $X \in |pow_{\sigma}(\mathcal{P})|$. *Then*

$$b\#X \quad \textit{implies} \quad \bigcap^{\#a}X = \bigcap^{\#b}(b\ a){\cdot}X.$$

As a corollary, $a\#\bigcap^{\#a}X$ *and* $supp(\bigcap^{\#a}X) \subseteq supp(X)\setminus\{a\}$.

Proof. The corollary follows by part 3 of Corollary 2.1.10 and by Theorem 2.3.1. For the first part, we reason as follows:

$$\begin{array}{rll}\bigcap^{\#a}X &= \bigcap\{X[a{\mapsto}u] \mid u{\in}|\mathcal{P}^{\partial}|\} & \text{Definition 5.2.1}\\ &= \bigcap\{((b\ a){\cdot}X)[b{\mapsto}u] \mid u{\in}|\mathcal{P}^{\partial}|\} & \text{Lemma 3.4.3}\\ &= \bigcap^{\#b}(b\ a){\cdot}X & \text{Definition 5.2.1}\end{array}$$

□

PROPOSITION 5.2.6. *If* $X \in |pow_{\sigma}(\mathcal{P})|$ *then* $\bigcap^{\#a}X \in |pow_{\sigma}(\mathcal{P})|$.

Proof. Finite support is Lemma 5.2.5. It remains to check conditions 1 and 2 of Definition 3.4.5:

(1) Suppose b is fresh (so $b\#X$) and suppose $v \in |\mathcal{P}^{\partial}|$. Using Lemma 5.2.5 suppose without loss of generality that $a\#v$. Then we reason as follows:

$$\begin{array}{rll}(\bigcap^{\#a}X)[b{\mapsto}v] &= \bigcap^{\#a}(X[b{\mapsto}v]) & \text{Lemma 5.2.3}\\ &= \bigcap^{\#a}X & \text{C 1 of Def 3.4.5},\ b\#X\end{array}$$

(2) Suppose b' is fresh (so $b'\#X$). Then we reason as follows:

$$\begin{array}{rll}(\bigcap^{\#a}X)[b{\mapsto}b'] &= \bigcap^{\#a}(X[b{\mapsto}b']) & \text{Lemma 5.2.3}\\ &= \bigcap^{\#a}((b'\ b){\cdot}X) & \text{C 2 of Def 3.4.5},\ b'\#X\\ &= (b'\ b){\cdot}(\bigcap^{\#a}X) & \text{Theorem 2.3.1}\end{array}$$

□

COROLLARY 5.2.7. *Suppose* $\mathcal{X} \in |pow_{\sigma}(\mathcal{P})|$. *Then* $\bigcap^{\#a}X$ *is equal to* $\bigwedge^{\#a}X$ *in* $pow_{\sigma}(\mathcal{P})$ *considered as a nominal poset (Definition 4.1.1) under subset inclusion.*

That is, $\bigcap^{\#a}X$ *is the largest element of* $\{Z \in |pow_{\sigma}(\mathcal{P})| \mid Z \subseteq X \wedge a\#Z\}$, *and not just the largest element of* $\{Z \in |pow(\mathcal{P})| \mid Z \subseteq X \wedge a\#Z\}$.

Proof. By Proposition 5.2.6 $\bigcap^{\#a}X$ is in $|pow_{\sigma}(\mathcal{P})|$, which is a nominal poset. The result follows. □

Because of Corollary 5.2.7, the set $\bigcap^{\#a}X$ from Definition 5.2.1 can be characterised in several ways:

PROPOSITION 5.2.8. *The following sets all exist and are equal to* $\bigcap^{\#a}X$ *in* $pow_{\sigma}(\mathcal{P})$:

$$\begin{array}{ll}\bigcup\{X'{\in}|pow_{\sigma}(\mathcal{P})| \mid X'{\subseteq}X,\ a\#X'\} & \bigcup\{X'{\in}|pow_{\sigma}(\mathcal{P})| \mid X'{\subseteq}X,\ supp(X'){\subseteq}supp(X)\setminus\{a\}\}\\ \bigcap\{X[a{\mapsto}u] \mid u{\in}|\mathcal{P}^{\partial}|\} & \bigcap\{X[a{\mapsto}n] \mid n{\in}\mathbb{A}\}\\ \{p \mid \forall u{\in}|\mathcal{P}^{\partial}|.\mathsf{N}c.p[u{\leftarrow}c] \in (c\ a){\cdot}X\} & \{p \mid \forall n{\in}\mathbb{A}.\mathsf{N}c.p[n{\leftarrow}c] \in (c\ a){\cdot}X\}\\ \bigcap\{X[a{\mapsto}n] \mid n{\notin}supp(X)\setminus\{a\}\} & \bigcap\{(n\ a){\cdot}X \mid n{\notin}supp(X)\setminus\{a\}\}.\end{array}$$

Above, n *ranges over all atoms, and may be equal to* a.

Proof. We consider each line in turn:

(1) This is Definition 5.2.1 and Corollaries 5.2.7 and 5.2.2.
(2) This is Propositions 4.3.5 and 4.3.6.
(3) This is the previous case combined with Proposition 3.4.2.

(4) This is Proposition 4.2.7. □

Remark 5.2.9. Proposition 5.2.8 converts a fresh-finite limit into a universal quantification 'for all $u \in |\mathcal{P}^{\partial}|$'—and then into a universal quantification 'for all atoms' (line 2) and finally into a И-quantifier 'for all fresh atoms' (line 4), both of which can be easier to handle than a universal quantification.

This is a theme of the nominal treatment of quantification; we saw it in Proposition 4.3.6, and we will see it again in Lemma 6.1.14.

This matters. Later on when we consider morphisms, we will need to know that a map $g^{\text{-}1}$ in Proposition 8.2.2 commutes with $\bigcap^{\#a}$. But $g^{\text{-}1}$ can change the underlying termlike σ-algebra—that is, the domain of quantification can change. That would on the face of it interfere with the universal quantifier, because extra elements might appear that make a true universal quantification into a false one.

Proposition 5.2.8 says this cannot happen—or, to express the same mathematical fact in a different language, it makes formal that the notion of morphism used in Proposition 8.2.2 has to take any extra elements into account. In the language of [Sel02], Proposition 5.2.8 implies that our models are *well-pointed*.

A dedicated discussion of such issues in the context of nominal models of the λ-calculus is in [GM11, Subsection 3.4]. Also, compare this discussion with the definition of *filter* below in Definition 6.1.1.

Theorem 5.2.10. *Suppose $\mathcal{P}$ is an $\mho$-algebra. Then the σ-algebra $pow_{\sigma}(\mathcal{P})$ from Definition 3.4.5 naturally becomes a nominal distributive lattice with $\forall$ where $\top$, $\wedge$, $\bot$, $\vee$, and $\forall$ are interpreted as $|\mathcal{P}|$, set intersection $\cap$, the empty set $\varnothing$, set union $\cup$, and $\bigcap^{\#a}$.*

Proof. By Lemma 5.1.2 for most of the connectives, and by Corollary 5.2.7 for $\forall$. □

This completes Part I. So far, we have defined nominal distributive lattices with $\forall$ and seen how to build them using nominal powersets. In Part II we show how to go from topologies (i.e. subsets of powersets subject to various sanity conditions, since powersets are usually very large) back to nominal distributive lattices with $\forall$.

II. NOMINAL SPECTRAL SPACE REPRESENTATION

6. COMPLETENESS

The key definitions of this section are that of *filter* in Definition 6.1.1, and of an *impredicative* nominal distributive lattice with $\forall$ in Definition 4.5.1. The main result is a representation result, Theorem 6.4.4, which represents an impredicative nominal distributive lattice with $\forall$ as a set of sets of prime filters.

The key technical results are in the sequence Lemma 6.1.10, Proposition 6.1.12, and Theorem 6.1.13, which use Zorn's lemma to exhibit every filter as a subset of some prime filter.

This story is familiar: it is standard to represent a lattice using sets of prime filters. However, making this work is not trivial. It is interesting to highlight three reasons for this:

(1) We must account for $\forall$, of course.
    This is extra structure and our treatment uses properties of nominal sets and the И-quantifier in subtle ways. See the discussion opening Subsection 6.1.
(2) Zorn's Lemma is related to the Axiom of Choice, which can cause difficulties with nominal sets because it may lead to non-finite support (our definition of a nominal set from Definition 2.1.5 requires finite support).
    We will find ourselves using elements which do have a permutation action—we are still within nominal techniques—but the elements do not necessarily have finite support. This is unusual. See Remark 6.1.3.
(3) Once these difficulties are navigated, we must still give points (prime filters) an $\mho$-algebra structure. There is no reason to expect prime filters to behave well and support an $\mho$-algebra structure. 'By magic', it just works: see Lemma 6.2.1.

A final technical hurdle is generated by our intended application of giving semantics to the untyped λ-calculus. In effect, this means that we want to consider lattices with a substitution action over themselves, in a suitable sense, whence the notion of *impredicativity* developed in Subsection 4.5. As usual for impredicative definitions, care is needed. Yet, once these definitions and results are in place, the main result Theorem 6.4.4 becomes quite natural.

6.1. Filters and prime filters

For this subsection, fix $\mathcal{D} \in \mathsf{nDi}\forall$ a nominal distributive lattice with $\forall$ (Definition 4.4.3). Recall from Definition 4.4.3 that $\mathcal{D}$ is a set with a finitely-supported permutation action, a σ-action (like a substitution action but abstractly specified as a nominal algebra), finite joins, fresh-finite meets, and satisfying a generalisation of the usual distributivity properties for lattices.

We start by defining our notion of (prime) filter, and proving that every filter is included in some prime filter.

The main definition is Definition 6.1.1 and the main result is Theorem 6.1.13. The main technical result is Lemma 6.1.10.

Condition 4 of Definition 6.1.1 is specific to the *nominal* filters. See also its verification in Lemma 6.1.10, in which $\forall$ is decomposed into И and the permutation action π—echoing Propositions 4.3.6 and 5.2.8. This decomposition of $\forall$ into И and π is important for two reasons:

— it converts an infinite conjunction over the entire domain into a И-quantified assertion—the И-quantifier has some excellent properties, such as commuting with conjunction *and* disjunction—and
— it does not depend on $\mathcal{D}^{\partial}$.

So condition 4 of Definition 6.1.1 means that to check the universal quantifier $\forall a.x$ we do not need to know about all of the programs of $\mathcal{D}$. We just need to know about the atoms, and in particular, we just need to know about the fresh atoms.

This is familiar from proof-theory. To prove $\Gamma \vdash \forall x.\phi$ we do not need to check $\phi[a\mapsto t]$ for every term t; we just check $\phi[x\mapsto y]$ for fresh y.

Definition 6.1.1. A **filter** in $\mathcal{D}$ is a nonempty subset $p \subseteq |\mathcal{D}|$ (which need not have finite support) such that:

(1) $\bot \notin p$ (we say p is **consistent**).
(2) If $x \in p$ and $x \leq x'$ then $x' \in p$ (we call p **up-closed**).
(3) If $x \in p$ and $x' \in p$ then $x \wedge x' \in p$.
(4) If $\text{И}b.(b\ a)\cdot x \in p$ then $\forall a.x \in p$.

The notion of *prime filter* is in Definition 6.1.11, and has no further surprises.

Remark 6.1.2. Condition 4 of Definition 6.1.1 seems odd: surely this should be

$$\forall u{\in}\mathcal{D}^{\partial}.x[a\mapsto u] \in p \text{ implies } \forall a.x \in p?$$

Or, in view of line 2 of Proposition 5.2.8 should it not be at least

$$\forall n{\in}\mathbb{A}.x[a\mapsto n_{\mathcal{D}^{\partial}}] \in p \text{ implies } \forall a.x \in p?$$

In fact, we shall see in Lemma 6.1.14 that condition 4 as written, implies both of these.

Remark 6.1.3. We do not assume that p has finite support, so the b bound by the И-quantifier in condition 4 of Definition 6.1.1 need not necessarily be fresh for p. So the reader familiar with nominal techniques should note that our use of И is atypical. The 'standard' decomposition of И into '$\forall$+freshness' and '$\exists$+freshness' familiar from e.g. Theorem 2.17 of [Gab11b], Theorem 6.5 of [Gab11a], or Theorem 9.4.6 of [Gab01] will not necessarily work here, at least not with respect to p without finite support. Nevertheless, we have enough structure to obtain the results we need.

What is the case, is that $x \in |\mathcal{D}|$ is assumed to have finite support, and b will be fresh for x.

It will be important that we allow p to have non-finite support; without this, we could not use Zorn's lemma in Theorem 6.1.13.

A dual notion of Definition 6.1.1 will also be useful:

DEFINITION 6.1.4. An **ideal** in $\mathcal{D}$ is a subset $Z \subseteq |\mathcal{D}|$ (which need not have finite support) such that:

(1) $\top \notin Z$.
(2) If $x \in Z$ and $x' \leq x$ then $x' \in Z$ (we call Z **down-closed**).
(3) If $x \in Z$ and $x' \in Z$ then $x \vee x' \in Z$.

REMARK 6.1.5. Definition 6.1.4 is not a perfect dual to Definition 6.1.1: we do not have $\forall$. (Correspondingly, we assume that a universal quantifier exists in $\mathcal{D}$, but not an existential.) This will not be a problem.[22]

DEFINITION 6.1.6. If $x \in |\mathcal{D}|$ then define $x\uparrow$ and $x\downarrow$ by

$$\boxed{x\uparrow = \{y \mid x \leq y\} \qquad \text{and} \qquad x\downarrow = \{y \mid y \leq x\}.}$$

LEMMA 6.1.7.*(1) If $a\#z$ and $z \leq x$ then $z \leq \forall a.x$.*
(2) If $b\#z, x$ and $z \leq (b\ a)\cdot x$ then $z \leq \forall a.x$.
(3) If $b\#z, y, x$ and $z \leq y \vee ((b\ a)\cdot x)$ then $z \leq y \vee \forall a.x$.

Proof. We consider each part in turn:

(1) Suppose $a\#z$ and $z \leq x$. By Lemma 4.1.7 $\forall a.z \leq \forall a.x$ and since $a\#z$ we have $\forall a.z = z$.
(2) Suppose $b\#z, x$ and $z \leq (b\ a)\cdot x$. By part 1 of this result $z \leq \forall b.(b\ a)\cdot x$. We use Lemma 4.1.3.
(3) From part 2 of this result and condition 2 of distributivity (Definition 4.4.1).

□

COROLLARY 6.1.8.*— If $x \neq \bot$ then $x\uparrow$ from Definition 6.1.6 is a filter.*
— Conversely if $x \neq \top$ then $x\downarrow$ is an ideal.

Proof. It is routine to verify conditions 1 to 3 of Definition 6.1.1. Suppose $x \leq (b\ a)\cdot y$ for all but finitely many b. We take one particular $b\#x, y$ and use part 2 of Lemma 6.1.7.

The case of $x\downarrow$ is no harder. □

DEFINITION 6.1.9. Suppose $p \subseteq |\mathcal{D}|$. Then define:

$$p{+}y = \bigcup\{(x \wedge y)\uparrow \mid x \in p\}$$

So $z \in p{+}y$ when $x \wedge y \leq z$ for some $x \in p$.

LEMMA 6.1.10. *Suppose p is a filter in $\mathcal{D}$ and suppose $y \in |\mathcal{D}|$. Then:*

— $p \subseteq p{+}y$.
— $y \in p{+}y$.
— $p{+}y$ is closed under conditions 2 to 4 of Definition 6.1.1 (so that if $p{+}y$ is consistent then it is a filter).

Proof. The first two parts are clear. We now check that $p{+}y$ satisfies conditions 2 to 4 of Definition 6.1.1:

(2) *If $z \in p{+}y$ and $z \leq z'$ then $z' \in p{+}y$.* By construction.
(3) *If $z \in p{+}y$ and $z' \in p{+}y$ then $z \wedge z' \in p{+}y$.* Suppose $z \geq x \wedge y$ and $z' \geq x' \wedge y$ for $x, x' \in p$. Then by condition 3 of Definition 6.1.1 $x \wedge x' \in p$, and it is a fact that $z \wedge z' \geq (x \wedge x') \wedge y$.

[22] We need $\wedge$ and $\vee$ to build filters. Later on when we model the untyped λ-calculus in Subsection 10.2, we will need $\forall$, $-\!\!\bullet$, and $\bullet$. We will not need an existential $\exists$. The existential may still exist; see Appendix B.2.

(4) *If* $\text{И}b.((b\ a)\cdot z \in p{+}y)$ *then* $\forall a.z \in p{+}y$. Suppose for all but finitely many b there exists an $x_b \in p$ such that $(b\ a)\cdot z \geq x_b \wedge y \in p$. Then there certainly exists some b such that $b\#y, z$ and $(b\ a)\cdot z \geq x_b \wedge y \in p$. We apply $\forall b$ to both sides and use distributivity (Definition 4.4.1), and we conclude that

$$\forall b.(b\ a)\cdot z \geq (\forall b.x_b)\wedge y \in p.$$

By Lemma 4.1.3 and condition 2 of Definition 6.1.1 we conclude that $\forall a.z \in p{+}y$ as required.

□

Recall from Definition 6.1.1 the notion of a *filter*:

DEFINITION 6.1.11.— Call a filter $p{\subseteq}|\mathcal{D}|$ **prime** when $x_1 \vee x_2 \in p$ implies either $x_1 \in p$ or $x_2 \in p$.
— Suppose p is a filter and $Z \subseteq |\mathcal{D}|$ is an ideal, and suppose $p \cap Z = \varnothing$. Call p **maximal with respect to** Z when for every filter p' with $p' \cap Z = \varnothing$, if $p \subseteq p'$ then $p = p'$.
— Call p **maximal** when it is maximal with respect to the ideal $\{\bot\}$.

PROPOSITION 6.1.12. *Suppose $p \subseteq |\mathcal{D}|$ is a filter and $Z \subseteq |\mathcal{D}|$ is an ideal, and suppose $p \cap Z = \varnothing$. If p is a maximal filter with respect to Z then it is prime.*

Proof. Suppose $y_1 \vee y_2 \in p$ and $y_1, y_2 \notin p$. By Lemma 6.1.10 and maximality we have that $(p{+}y_1) \cap Z \neq \varnothing$ and $(p{+}y_2) \cap Z \neq \varnothing$. It follows that there exist $x_1, x_2 \in p$ with $x_1 \wedge y_1, x_2 \wedge y_2 \in Z$. Since Z is an ideal,

$$(x_1 \wedge y_1) \vee (x_2 \wedge y_2) \in Z.$$

Now we rearrange the left-hand-side to deduce that

$$u = (x_1 \vee x_2) \wedge (x_1 \vee y_2) \wedge (y_1 \vee x_2) \wedge (y_1 \vee y_2) \in Z.$$

We now note that $x_1 \vee x_2 \in p$ (since $x_1 \in p$, and indeed also $x_2 \in p$) and $x_1 \vee y_2 \in p$ (since $x_1 \in p$) and $y_1 \vee x_2 \in p$ (since $x_2 \in p$) and $y_1 \vee y_2 \in p$ by assumption. But then $u \in p$, contradicting our assumption that $p \cap Z = \varnothing$. □

THEOREM 6.1.13. *Suppose $Z \subseteq |\mathcal{D}|$ is an ideal and $p \subseteq |\mathcal{D}|$ is a filter and $p \cap Z = \varnothing$. Then there exists a prime filter q with $p \subseteq q$ and $q \cap Z = \varnothing$.*

As a corollary, if p is a filter then there exists a prime filter q containing p.

Proof. The corollary follows taking Z to be $\{\bot\}$, which we can easily verify is an ideal. We now consider the main result.

If $\mathcal{C}$ is a chain in the set of filters ordered by subset inclusion, then $\bigcup \mathcal{C}$ is an upper bound for $\mathcal{C}$.[23] By Zorn's Lemma [End77, page 153] the set of filters p' such that $p \subseteq p'$ and $p' \cap Z = \varnothing$ has a maximal element q with respect to inclusion. By Proposition 6.1.12 q is prime. □

LEMMA 6.1.14. *The following conditions are equivalent (below, n ranges over all atoms, including a):*

$$\forall a.x \in p \quad\Leftrightarrow\quad \forall u{\in}|\mathcal{D}^{\partial}|.x[a{\mapsto}u] \in p \quad\Leftrightarrow\quad \forall n{\in}\mathbb{A}.x[a{\mapsto}n] \in p \quad\Leftrightarrow\quad \text{И}b.(b\ a)\cdot x \in p$$

Proof. Suppose $\forall a.x \in p$. By $(\forall{\leq})$ and condition 2 of Definition 6.1.1 also $x[a{\mapsto}u] \in p$ for every $u{\in}|\mathcal{D}^{\partial}|$.

It follows in particular that $x[a{\mapsto}n_{\mathcal{D}^{\partial}}] \in p$ for every $n \in \mathbb{A}$.

It also follows that $x[a{\mapsto}b] \in p$ for all $b\#x$ so by Lemma 3.2.3 also $\text{И}b.(b\ a)\cdot x \in p$.

Now suppose $\text{И}b.(b\ a)\cdot x \in p$. By condition 4 of Definition 6.1.1 also $\forall a.x \in p$. □

PROPOSITION 6.1.15. *Suppose p is a filter. Then:*

[23] We do *not* insist on finite support here.

(1) $\bot \notin p$ *and* $\top \in p$.
(2) $x \wedge y \in p$ *if and only if* $x \in p$ *and* $y \in p$.
(3) $\forall a.x \in p$ *if and only if* $\reflectbox{N} b.(b\ a)\cdot x \in p$.
(4) *If* p *is prime then* $x \vee y \in p$ *if and only if* $x \in p$ *or* $y \in p$.[24]

Proof. (1) The first part is condition 1 of Definition 6.1.1. For the second part, by assumption in Definition 6.1.1 p is nonempty, so there exists some $x \in p$. Now $\mathcal{D}$ has a top element $\top$, so $x \leq \top$ and by condition 2 of Definition 6.1.1 $\top \in p$.
(2) From conditions 2 and 3 of Definition 6.1.1.
(3) This is Lemma 6.1.14.
(4) Routine, again using condition 2 of Definition 6.1.1.

□

6.2. The amgis-action on (prime) filters

For this subsection, fix $\mathcal{D} \in \mathsf{nDi}\forall$ a nominal distributive lattice with $\forall$ (Definition 4.4.3).

Recall from Definition 3.3.1 the pointwise ᘁ-action $p[u\leftarrow\!\!\dashv a] = \{x \mid x[a\mapsto u] \in p\}$ where $u \in |\mathcal{D}^{\partial}|$. In this subsection we check that this action preserves the property of being a (prime) filter (Definitions 6.1.1 and 6.1.11).

The work happens in the key technical result Lemma 6.2.1; Proposition 6.2.3 then puts the result in a some nice packaging.

Lemma 6.2.1. *If* p *is a filter in* $\mathcal{D}$ *then so is* $p[u\leftarrow\!\!\dashv a]$. *Furthermore, if* p *is prime then so is* $p[u\leftarrow\!\!\dashv a]$.

Proof. We check the conditions in Definition 6.1.1. We use Proposition 3.3.2 without comment:

— $\bot \notin p[u\leftarrow\!\!\dashv a]$. Since by $(\sigma\#)$ $\bot[a\mapsto u] = \bot$.
— *If* $x \in p[u\leftarrow\!\!\dashv a]$ *and* $x \leq x'$ *then* $x' \in p[u\leftarrow\!\!\dashv a]$. From Lemma 4.3.2.
— *If* $x \in p[u\leftarrow\!\!\dashv a]$ *and* $x' \in p[u\leftarrow\!\!\dashv a]$ *then* $x \wedge x' \in p[u\leftarrow\!\!\dashv a]$. Since by assumption the σ-action is compatible, so $(x \wedge x')[a\mapsto u] = x[a\mapsto u] \wedge (x'[a\mapsto u])$ (Definition 4.3.1).
— *If* $\reflectbox{N} b'.((b'\ b)\cdot x \in p[u\leftarrow\!\!\dashv a])$ *then* $\forall b.x \in p[u\leftarrow\!\!\dashv a]$. Choose some fresh c (so $c\#x, u$). It is a fact that $x = (c\ b)\cdot((c\ b)\cdot x)$ and by $(\forall\alpha)$ also $\forall b.x = \forall c.(c\ b)\cdot x$. Thus, we may rename to assume without loss of generality that $b\#u$.
Now suppose $((b'\ b)\cdot x)[a\mapsto u] \in p$ for all but finitely many b'; so suppose $b'\#u$. By Corollary 2.1.10 $(b'\ b)\cdot u = u$. Thus by Remark 3.1.2 and Corollary 2.1.10 we have that $(b'\ b)\cdot(x[a\mapsto u]) \in p$ for all but finitely many b'. Therefore $\forall b.(x[a\mapsto u]) \in p$ and by compatibility (Definition 4.3.1) $(\forall b.x)[a\mapsto u] \in p$.

Now suppose p is prime and suppose $(y_1 \vee y_2)[a\mapsto u] \in p$. Then by compatibility (Definition 4.3.1) $y_1[a\mapsto u] \vee (y_2[a\mapsto u]) \in p$. Therefore either $y_1[a\mapsto u] \in p$ or $y_2[a\mapsto u] \in p$. □

Definition 6.2.2. If $\mathcal{D} \in \mathsf{nDi}\forall$ write $points(\mathcal{D})$ for the ᘁ-algebra determined by prime filters and the pointwise actions from Definition 3.3.1. That is:

— $|points(\mathcal{D})| = \{p \subseteq |\mathcal{D}| \mid p \text{ is a prime filter}\}$.
— $points(\mathcal{D})^{\partial} = \mathcal{D}^{\partial}$.
— $\pi\cdot p = \{\pi\cdot x \mid x \in p\}$ and $p[u\leftarrow\!\!\dashv a] = \{x \mid x[a\mapsto u] \in p\}$ for $u \in |points(\mathcal{D})^{\partial}|$.

Proposition 6.2.3. $points(\mathcal{D})$ *is indeed an* ᘁ*-algebra.*

Proof. This is just Lemma 6.2.1 combined with Proposition 3.3.4. □

[24] We assume that $x \in |\mathcal{D}|$ is finitely supported because in Definition 4.1.1 we assume $(|\mathcal{D}|, \cdot)$ is a nominal set. We do not assume that a filter p is finitely supported in Definition 6.1.1.
This means that we may assume that b is fresh for x under the $\reflectbox{N} b$ quantifier, but we do not know that b is fresh for p. This will not be a problem.

6.3. Injecting $\mathcal{D}$ into the set of sets of prime filters

Recall from Definition 3.4.5 the notion of *σ-powerset algebra pow_σ*, and from Definition 6.1.11 the notion of a *prime filter*.

In this subsection we consider how to embed $\mathcal{D}$ a nominal distributive lattice with $\forall$ in the σ-powerset of its prime filters. The main definition is Definition 6.3.1. The main results are Lemma 6.4.2, Corollary 6.3.3, and Lemma 6.3.5.

It is standard to embed a lattice into sets of prime filters, and Definition 6.3.1 is has the form that one would expect. There is extra structure; for instance Lemma 6.4.1 and part 3 of Lemma 6.4.2. With the results we have proven so far, we can deal with this extra structure.

Definition 6.3.1. Suppose $\mathcal{D}$ is a nominal distributive nominal lattice with $\forall$. Define

$$x^\bullet = \{p \text{ a prime filter} \mid x \in p\}.$$

Lemma 6.3.2. *$x \leq y$ if and only if $x^\bullet \subseteq y^\bullet$.*

Proof. Suppose $x \leq y$. By condition 2 of Definition 6.1.1 if $x \in p$ then $y \in p$. It follows that $x^\bullet \subseteq y^\bullet$.

Suppose $x \not\leq y$. By Corollary 6.1.8 and Theorem 6.1.13 there exists a prime filter p containing $y{\uparrow}$ (so containing y) and disjoint from $x{\downarrow}$ thus not containing x. Then $p \in y^\bullet$ and $p \notin x^\bullet$. □

Corollary 6.3.3. *The assignment $x \mapsto x^\bullet$ is injective.*

Proof. Direct from Lemma 6.3.2. □

Corollary 6.3.4. *$supp(x^\bullet) = supp(x)$.*

Proof. Using part 3 of Theorem 2.3.1 and Corollary 6.3.3; for more details see [Gab11a, Theorem 4.7]. □

Lemma 6.3.5 will be useful later but we prove it here because it has a family resemblance to Corollary 6.3.3 (see Remark 6.3.6). It expresses that "if all the ways of extending p to a prime filter q contain x, then p must already contain x":

Lemma 6.3.5. *Suppose $p \subseteq |\mathcal{D}|$ is a filter and $x \in |\mathcal{D}|$, and suppose for every prime filter r, if $p \subseteq r$ then $x \in r$. Then $x \in p$.*

Proof. Suppose $x \notin p$. By Corollary 6.1.8 $x{\downarrow}$ is an ideal and by Theorem 6.1.13 there exists a prime filter r such that $p \subseteq r$ and $x \notin r$. The result follows. □

Remark 6.3.6. A nice rephrasing of Lemma 6.3.5 is possible using Definition 6.3.1 (it will be useful later in Proposition 9.4.6):

$$p = \bigcap \{x^\bullet \mid x \in p\}.$$

6.4. The map from x, to prime filters containing x, as a morphism

For this subsection, fix $\mathcal{D} \in \mathsf{nDi}\forall$ a nominal distributive lattice with $\forall$ (Definition 4.4.3). Recall from Definition 6.3.1 that if $x \in |\mathcal{D}|$ then $x^\bullet$ is the set of prime filters in $\mathcal{D}$ that contain x.

By Proposition 6.2.3 $points(\mathcal{D})$ is an $\mho$-algebra. So following Definition 3.4.1, sets of points $X \subseteq |points(\mathcal{D})|$ inherit an action $X[a{\mapsto}u] = \{p \mid p[u{\leftarrow}a] \in X\}$. With this action we have the following:

Lemma 6.4.1. *Suppose $x \in |\mathcal{D}|$ and $u \in |\mathcal{D}^\partial|$. Then:*

(1) $\pi{\cdot}(x^\bullet) = (\pi{\cdot}x)^\bullet$
(2) $x^\bullet[a{\mapsto}u] = (x[a{\mapsto}u])^\bullet$

Proof. The case of $\pi{\cdot}(x^\bullet)$ is direct from Theorem 2.3.1 (a proof by concrete calculations similar to the case of $x^\bullet[a{\mapsto}u]$ is also possible). For the case of $x^\bullet[a{\mapsto}u]$, we reason as follows:

$$\begin{array}{r@{\ }l@{\qquad}l}
p \in (x^\bullet)[a{\mapsto}u] & \Leftrightarrow \text{И}c.p[u{\leftarrow}c] \in ((c\ a){\cdot}x)^\bullet & \text{Prop 3.4.2, pt 1 this result} \\
& \Leftrightarrow \text{И}c.(c\ a){\cdot}x \in p[u{\leftarrow}c] & \text{Definition 6.3.1} \\
& \Leftrightarrow \text{И}c.((c\ a){\cdot}x)[c{\mapsto}u] \in p & \text{Proposition 3.3.2} \\
& \Leftrightarrow p \in (x[a{\mapsto}u])^\bullet & (\sigma\alpha),\ \text{Definition 6.3.1}
\end{array}$$

□

Lemma 6.4.2.*(1)* $\bot^\bullet = \varnothing$ *and* $\top^\bullet = |points(\mathcal{D})|$
(2) $(x{\wedge}y)^\bullet = x^\bullet \cap y^\bullet$
(3) $(\forall a.x)^\bullet = \bigcap^{\#a}(x^\bullet)$ *(Definition 5.2.1)*
(4) $(x{\vee}y)^\bullet = x^\bullet \cup y^\bullet$

Proof. Parts 1, 2, and 4 just reformulate parts 1, 2, and 4 of Proposition 6.1.15.

For part 3, suppose $p \in (\forall a.x)^\bullet$. By Definition 6.3.1 this is if and only if $\forall a.x \in p$. By Lemma 6.1.14 this is if and only if $x[a{\mapsto}n] \in p$ for every $n{\in}\mathbb{A}$. Using Definition 6.3.1 and Lemma 6.4.1 this is if and only if $p \in x^\bullet[a{\mapsto}n]$ for every $n{\in}\mathbb{A}$. We use line 2 of Proposition 5.2.8. □

Definition 6.4.3. Define $\mathcal{D}^\bullet \in \mathsf{nDi}\forall$ a nominal distributive lattice with $\forall$ by the following data:

(1) $|\mathcal{D}^\bullet| = \{x^\bullet \mid x \in |\mathcal{D}|\}$.
(2) $(\mathcal{D}^\bullet)^\partial = \mathcal{D}^\partial$.
(3) $\mathcal{D}^\bullet$ has permutation and σ-actions following Definition 3.4.1 (so $\pi{\cdot}x^\bullet = \{\pi{\cdot}p \mid p \in x^\bullet\}$ and $x^\bullet[a{\mapsto}u] = \{p \mid p[u{\leftarrow}a] \in x^\bullet\}$).

If $\mathcal{D}$ is impredicative (Definition 4.5.1), so we assume a σ-algebra morphism $\partial : \mathcal{D}^\partial \to \mathcal{D}$, then $\mathcal{D}^\bullet$ naturally becomes impredicative where:

— $\partial_{\mathcal{D}^\bullet} u = (\partial_{\mathcal{D}} u)^\bullet$ for $u \in |\mathcal{D}^\partial| = |(\mathcal{D}^\bullet)^\partial|$.

It is now easy to state and prove a nominal sets representation theorem, representing an abstract $\mathcal{D} \in \mathsf{inDi}\forall$ concretely as the nominal sets-based structure $\mathcal{D}^\bullet$:

Theorem 6.4.4 (First representation theorem). *If $\mathcal{D}$ is in* $\mathsf{nDi}\forall$ *then so is $\mathcal{D}^\bullet$, and the pair of maps $(x \mapsto x^\bullet, u \mapsto u)$ is an isomorphism from $\mathcal{D}$ to $\mathcal{D}^\bullet$ in* $\mathsf{nDi}\forall$.
If furthermore $\mathcal{D}$ is in $\mathsf{inDi}\forall$ *(is impredicative) then so is $\mathcal{D}^\bullet$ and $x \mapsto x^\bullet$ is an isomorphism from $\mathcal{D}$ to $\mathcal{D}^\bullet$.*

Proof. We unpack Definition 6.4.3 and use Lemmas 6.4.1 and 6.4.2 to check the conditions on morphisms from Definition 4.4.5. Surjectivity is by construction and injectivity is by Corollary 6.3.3. □

Theorem 9.4.11 extends Theorem 6.4.4 with $\bullet$ and $-\!\!\bullet$.

7. NOMINAL σ-TOPOLOGICAL SPACES

7.1. The basic definition

Definition 7.1.1. A **nominal σ-topological space** $\mathcal{T}$ is a tuple $(|\mathcal{T}|, \cdot, \mathcal{T}^\partial, \sigma, opens(\mathcal{T}))$ where

— $(|\mathcal{T}|, \cdot, \mathcal{T}^\partial, \sigma)$ forms an σ-algebra (Definition 3.2.1) and
— $opens(\mathcal{T}) {\subseteq} |pow(\mathcal{T})|$ (i.e. a set of finitely supported sets; see Subsection 2.4.1) is a set of **open sets**.

For clarity, we explicitly unpack what $opens(\mathcal{T}) {\subseteq} |pow(\mathcal{T})|$ (Definition 3.4.5) means for $X \in opens(\mathcal{T})$:

(i) X must have finite support.
(ii) $a\#X$ must imply that $X[a{\mapsto}u] = X$.

(iii) $b\#X$ must imply that $X[a\mapsto b] = (b\ a)\cdot X$.

Furthermore we impose the following conditions on $opens(\mathcal{T})$:

(1) If $X \in opens(\mathcal{T})$ and $u \in |F(\mathcal{D})^{\partial}|$ then $\pi\cdot X \in opens(\mathcal{T})$ and $X[a\mapsto u] \in opens(\mathcal{T})$ (where $\pi\cdot X$ and $X[a\mapsto u]$ are the pointwise actions from Definition 3.4.1).
(2) $\emptyset \in opens(\mathcal{T})$ and $|\mathcal{T}| \in opens(\mathcal{T})$
(3) If $X \in opens(\mathcal{T})$ and $Y \in opens(\mathcal{T})$ then $X \cap Y \in opens(\mathcal{T})$.
(4) If $\mathcal{X} \subseteq opens(\mathcal{T})$ is strictly finitely supported (Definition 2.4.2) then $\bigcup \mathcal{X} \in opens(\mathcal{T})$; we call this a **strictly finitely supported union** of open sets.

Remark 7.1.2. As mentioned above, $\pi\cdot X$ and $X[a\mapsto u]$ mentioned in Definition 7.1.1 are the *pointwise* actions from Definition 3.4.1.

These are inherited from the permutation action and ꭥ-action on the underlying points, using the fact that open sets are sets of points and points are delivered to us in Definition 7.1.1 as an ꭥ-algebra. We spell this out for the reader's convenience:

$$\pi\cdot X = \{\pi\cdot t \mid t \in X\} \qquad X[a\mapsto u] = \{t \mid t[u\mapsfrom a] \in X\} \quad \text{where } u \in |\mathcal{T}^{\partial}|$$

Remark 7.1.3. Topological spaces over Zermelo-Fraenkel (**ZF**) sets—that is, over 'ordinary' sets—are such that an arbitrary union of open sets is open.

Condition 4 of Definition 7.1.1 generalises that condition, because any ZF set is also naturally an FM set with the trivial permutation action, and with empty support; so arbitrary sets of ZF sets are already strictly finitely supported, by $\varnothing$.

There are *two* natural generalisations of the topological condition to the nominal case: we could insist on finitely supported unions, or on strictly finitely supported unions. We used the first option in topological representations of first-order logic in [Gab12; Gab11b] and of the И-quantifier in [GLP11].

In this paper, *strict* finite support seems to be required—we have not checked whether [Gab12; Gab11b; GLP11] would be re-done with this stricter condition. To see where it is used, the reader can search this paper for uses of Lemma 2.4.3, starting with the proof of Theorem 7.2.2.

Remark 7.1.4. The reader might be surprised that Definition 7.1.1 makes no mention of $\bigcap^{\#a}$. Surely we should insist that if $U \in opens(\mathcal{T})$ then $\bigcap^{\#a} U \in opens(\mathcal{T})$?

We could do this and no harm would come of it. However, we can also leave $\bigcap^{\#a}$ to later, when we introduce *coherence* in Definition 7.4.1. We prefer this because it postpones complexity and increases the generality of the mathematics between now and then.

7.2. The map F from distributive lattices to nominal σ-topological spaces

Definition 7.2.1. Suppose $\mathcal{D} \in \mathsf{nDi}\forall$ is a nominal distributive lattice with $\forall$.

Define $F(\mathcal{D})$ a nominal σ-topological space (Definition 6.4.3) by (technical references follow):

(1) $F(\mathcal{D})$ has as underlying σ-algebra $points(\mathcal{D})$ from Definition 6.2.2, so that $|F(\mathcal{D})| = |points(\mathcal{D})|$ and $F(\mathcal{D})^{\partial} = \mathcal{D}^{\partial}$, and $\pi\cdot p = \{\pi\cdot x \mid x \in p\}$ and $p[u\mapsfrom a] = \{x \mid x[a\mapsto u] \in p\}$ for $u \in |\mathcal{D}^{\partial}|$.
(2) The topology $opens(F(\mathcal{D}))$ is generated under strictly finitely supported unions by $\{x^{\bullet} \mid x \in |\mathcal{D}|\}$.

References for technical definitions above: $points(\mathcal{D})$ is from Definition 6.2.2, and the pointwise actions are from Definition 3.4.1. Strict finite support is from Definition 2.4.2 and $x^{\bullet}$ is from Definition 6.3.1.

So $X \in opens(F(\mathcal{D}))$ when $X = \bigcup \mathcal{X}$ where $\mathcal{X} = \{x_i^{\bullet} \mid i \in I\}$ is a strictly finitely supported set of sets of points of the form $x^{\bullet}$.

Theorem 7.2.2. *If $\mathcal{D} \in \mathsf{nDi}\forall$ is a nominal distributive lattice (Definition 4.5.1) then $F(\mathcal{D})$ is a σ-topological space.*

Proof. We consider the properties in Definition 7.1.1 in turn:

(i) By assumption every $x \in |\mathcal{D}|$ has finite support, so by Corollary 6.3.4 and Lemma 2.4.3, so does every $X \in \mathit{opens}(F(\mathcal{D}))$.

(ii) *If $a\#X$ then $X[a{\mapsto}u] = X$.* Suppose $a\#X$. By assumption $X = \bigcup x_i^\bullet$ for some strictly finitely supported union of $x_i^\bullet$. By Lemma 2.4.3 and Corollary 6.3.4 $a\#x_i$ for every i so by $(\sigma\#)$ $x_i[a{\mapsto}u] = x_i$. By Lemma 6.4.1 $(x_i[a{\mapsto}u])^\bullet = x_i^\bullet$. We use part 4 of Lemma 6.4.2.

(iii) *If $b\#X$ then $X[a{\mapsto}u] = (b\ a)\cdot X$.* Suppose $b\#X$. By assumption $X = \bigcup x_i^\bullet$ for some strictly finitely supported union of $x_i^\bullet$. By Lemma 2.4.3 and Corollary 6.3.4 $b\#x_i$ for every i so by Lemma 3.2.3 $x_i[a{\mapsto}b] = (b\ a)\cdot x_i$. By Lemma 6.4.1 $((b\ a)\cdot x_i)^\bullet = (b\ a)\cdot x_i^\bullet$. We use part 4 of Lemma 6.4.2.

Furthermore:

(1) *If $X \in \mathit{opens}(F(\mathcal{D}))$ and $u \in |F(\mathcal{D})^\partial|$ then $\pi\cdot X \in \mathit{opens}(F(\mathcal{D}))$ and $X[a{\mapsto}u] \in \mathit{opens}(F(\mathcal{D}))$.* The case of π is direct from Theorem 2.3.1.
For the case of $[a{\mapsto}u]$ we note that by definition X is equal to some strictly finitely-supported union $\bigcup_{i\in I} x_i^\bullet$. We then use Lemma 5.1.1 and part 3 of Definition 6.4.3.

(2) $\varnothing$ is open by construction and $\top^\bullet = |\mathit{points}(\mathcal{D})|$.

(3) Suppose $X, Y \in \mathit{opens}(F(\mathcal{D}))$. So $X = \bigcup x_i^\bullet$ and $Y = \bigcup y_j^\bullet$ for some strictly finitely supported sets $\{x_i \mid i \in I\}, \{y_j \mid j \in J\} \subseteq |\mathcal{D}|$. By Theorem 2.3.1 $\mathit{supp}(x_i \wedge y_j) \subseteq \mathit{supp}(x_i) \cup \mathit{supp}(y_j)$. Thus using part 2 of Lemma 6.4.2 and some elementary sets calculations $X \cap Y = \bigcup_{i\in I, j\in J} (x_i \wedge y_j)^\bullet$ and this is a strictly finitely supported union.

(4) *If $\mathcal{X} \subseteq \mathit{opens}(F(\mathcal{D}))$ is strictly finitely supported then $\bigcup\mathcal{X} \in \mathit{opens}(F(\mathcal{D}))$.* Using Corollary 2.4.5.

□

Remark 7.2.3. Notice that we use Lemma 2.4.3 to check each of the three conditions imposed on open sets in Theorem 7.2.2. This is where we use the 'closure under *strict* finite support' in part 4 of Definition 7.2.1.

7.3. Compactness

7.3.1. The definition, and general properties. Fix $\mathcal{T}$ a nominal σ-topological space.

Definition 7.3.1. Suppose $\mathcal{U} \subseteq \mathit{opens}(\mathcal{T})$ and $U \in \mathit{opens}(\mathcal{T})$.

— Say $\mathcal{U}$ **covers** U when $\mathcal{U}$ is strictly finitely supported and $U \subseteq \bigcup\mathcal{U}$. Call $\mathcal{U}$ a **cover** when it covers $|\mathcal{T}|$.
— Call U **compact** when every cover of U has a finite subcover. Write $\mathit{cpct}(\mathcal{T})$ for the set of compact open sets of $\mathcal{T}$:

$$\mathit{cpct}(\mathcal{T}) = \{U \in \mathit{opens}(\mathcal{T}) \mid U \text{ is compact}\}$$

Proposition 7.3.2 looks familiar enough, but we are in a nominal context so we have to check facts about support. It all works:

Proposition 7.3.2. *Suppose $U, V \in \mathit{cpct}(\mathcal{T})$. Then:*

(1) $\varnothing$ is compact.
(2) $U \cup V$ is compact.
(3) $\pi\cdot U$ is compact.

Proof. (1) There are two covers of $\varnothing$: the empty set of open sets and the set containing the empty set of points. Both are finite.

(2) Suppose U and V are compact and $\mathcal{W}$ covers $U \cup V$. Then $\{W \cap U \mid W \in \mathcal{W}\}$ covers U. By Lemma 2.4.3 and Theorem 2.3.1 $\mathit{supp}(W \cap U) \subseteq \mathit{supp}(\mathcal{W}) \cup \mathit{supp}(U)$ for each $W \in \mathcal{W}$. Thus

$\{W \cap U \mid W \in \mathcal{W}\}$ covers U and is strictly supported, and we obtain a finite subcover of U. Reasoning similarly for $\{W \cap V \mid W \in \mathcal{W}\}$ we obtain a finite subcover of V.
Putting these two finite subcovers together, we obtain one of $U \cup V$.

(3) Using Theorem 2.3.1 and Proposition 2.3.3.

□

In a sense, Definition 7.4.1 is a continuation of Proposition 7.3.2.

7.3.2. The specific case of $F(\mathcal{D})$. Fix some $\mathcal{D} \in \mathsf{nDi}\forall$.

LEMMA 7.3.3. *Suppose $\mathcal{U}$ is a strictly finitely supported set of open sets in $F(\mathcal{D})$. Then*

$$\bigcup \mathcal{U} = \bigcup \{x^\bullet \mid supp(x) \subseteq supp(\mathcal{U}) \land \exists U \in \mathcal{U}. x^\bullet \subseteq U\}.$$

Proof. By construction in Definition 7.2.1 every $U \in \mathcal{U}$ is a strictly finitely supported union $\bigcup_{i \in I} x^\bullet_{U,i}$. By Lemma 2.4.3 and Corollary 6.3.4 $supp(x_{U,i}) \subseteq supp(U)$ for every $x_{U,i}$. Similarly $\mathcal{U}$ is strictly finitely supported so by Lemma 2.4.3 $supp(U) \subseteq supp(\mathcal{U})$. The result follows. □

LEMMA 7.3.4. *Suppose $\mathcal{U} \subseteq cpct(F(\mathcal{D}))$ and $U \subseteq |\mathcal{D}|$. Then if $\mathcal{U}$ covers U then so does*

$$\mathcal{Y} = \{y^\bullet \mid supp(y) \subseteq supp(\mathcal{U}) \land \exists U \in \mathcal{U}. y^\bullet \subseteq U\}.$$

Proof. From Lemma 7.3.3 $\mathcal{Y}$ covers U. □

DEFINITION 7.3.5. If $\mathcal{Y} \subseteq cpct(F(\mathcal{T}))$ then define $\mathcal{Y}^{\bullet\text{-}1} \subseteq |\mathcal{D}|$ by

$$\mathcal{Y}^{\bullet\text{-}1} = \{y \in |\mathcal{D}| \mid y^\bullet \in \mathcal{Y}\}.$$

LEMMA 7.3.6. *Continuing the notation of Definition 7.3.5, $x^\bullet \in \mathcal{Y}$ if and only if $x \in \mathcal{Y}^{\bullet\text{-}1}$.*

DEFINITION 7.3.7. Given $X \subseteq |\mathcal{D}|$ write $X_{\vee\downarrow}$ for the least set such that:

— If $x \in X$ then $x \in X_{\vee\downarrow}$.
— If $x', x \in X_{\vee\downarrow}$ then $x' \vee x \in X_{\vee\downarrow}$.
— If $x \in X_{\vee\downarrow}$ and $x' \leq x$ then $x' \in X_{\vee\downarrow}$.

Similarly write $X_{\wedge\uparrow}$ for the least set such that:

— If $x \in X$ then $x \in X_{\wedge\uparrow}$.
— If $x', x \in X_{\wedge\uparrow}$ then $x' \wedge x \in X_{\wedge\uparrow}$.
— If $x \in X_{\wedge\uparrow}$ and $x \leq x'$ then $x' \in X_{\wedge\uparrow}$.

Intuitively, $X_{\vee\downarrow}$ is trying to be the least ideal (Definition 6.1.4) containing X; it may fail to be an ideal if $\top \in X_{\vee\downarrow}$. Similarly, $X_{\wedge\uparrow}$ is moving in the direction of being a filter (Definition 6.1.1) containing X; it may fail to be a filter if either $\bot \in X_{\wedge\uparrow}$, or $\text{И}b.(b\ a) \cdot x \in X_{\wedge\uparrow}$ and $\forall a. x \notin X_{\wedge\uparrow}$.

PROPOSITION 7.3.8. *Suppose $z \in |\mathcal{D}|$. Then $z^\bullet$ is open and compact in $F(\mathcal{D})$.*

Proof. $z^\bullet$ is open by construction in Definition 7.2.1.

Now consider a cover $\mathcal{Y}$ of $z^\bullet$. By Lemma 7.3.4 we may assume without loss of generality that every element of $\mathcal{Y}$ has the form $y^\bullet$ for some $x \in |\mathcal{D}|$. Write

$$X = \{x \mid \exists y \in \mathcal{Y}^{\bullet\text{-}1}. (z \leq y \vee x)\}_{\wedge\uparrow}.$$

Recall from Definitions 7.3.5 and 7.3.7 that $\mathcal{Y}^{\bullet\text{-}1}$ 'strips the -$^\bullet$ from the elements of $\mathcal{Y}$' and -$_{\wedge\uparrow}$ 'tries to make a filter' out of its argument.

We observe using Lemma 7.3.6 and some routine calculations that $\mathcal{Y}$ has a finite subcover if and only if $\bot \in X$.

We now observe that X satisfies conditions 2, 3, and 4 of Definition 6.1.1:

(2) *X is up-closed.* By the use of -$_{\wedge\uparrow}$.
(3) *$x \in X$ and $x' \in X$ imply $x \wedge x' \in X$.* By the use of -$_{\wedge\uparrow}$.

(4) *If* $\text{И}b.((b\ a)\cdot x \in X)$ *then* $\forall a.x \in X$. Choose fresh b (so $b\#z, x, X, \mathcal{Y}$), so there exist $y_i \in \mathcal{Y}^{\bullet\text{-}1}$ for $1 \leq i \leq n$ and $x_1, \ldots, x_n$ such that $z \leq y_i \vee x_i$ for $1 \leq i \leq n$, and $(b\ a)\cdot x \geq x_1 \wedge \ldots \wedge x_n$.
Now $b\#\mathcal{Y}$ and the cover $\mathcal{Y}$ is assumed to be strictly finitely supported, so by Lemma 2.4.3 and Corollary 6.3.4 also $b\#y_i$ for $1 \leq i \leq n$. By part 3 of Lemma 6.1.7 $z \leq y_i \vee \forall b.x_i$ for $1 \leq i \leq n$.
Also, using Lemma 4.1.5 $\forall b.(b\ a)\cdot x \geq \forall b.x_1 \wedge \ldots \forall b.x_n$.
It follows using Lemma 4.1.3 that $\forall a.x \in X$.

It is now useful to consider two distinct possibilities: $X \cap (\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow} \neq \varnothing$ or $X \cap (\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow} = \varnothing$. We treat each in turn:

— *Suppose* $X \cap (\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow} \neq \varnothing$. So take any $x \in X \cap (\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow}$.
Since $x \in (\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow}$, there exist $y'_1, \ldots, y'_m \in \mathcal{Y}^{\bullet\text{-}1}$ with $x \leq y'_1 \vee \ldots \vee y'_m$.
Since $x \in X$, there exist $y_1, \ldots, y_n \in \mathcal{Y}^{\bullet\text{-}1}$ and $x_1, \ldots, x_n \in X$ such that $z \leq y_i \vee x_i$ for $1 \leq i \leq n$ and $x \geq x_1 \wedge \ldots \wedge x_n$.
We note that $z \leq (y_1 \vee x_1) \wedge \ldots \wedge (y_n \vee x_n)$ and by some easy calculations that

$$z \leq x \vee y_1 \vee \ldots \vee y_n \leq y'_1 \vee \ldots \vee y'_m \vee y_1 \vee \ldots \vee y_n.$$

Thus, using part 4 of Lemma 6.4.2 and Lemma 6.3.2 we have that $\{(y'_1)^\bullet, \ldots, (y'_m)^\bullet, y_1^\bullet, \ldots, y_n^\bullet\} \subseteq \mathcal{Y}$ covers $z^\bullet$.
— *Now suppose* $X \cap (\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow} = \varnothing$. In particular $\bot \notin X$, so by the arguments above X is a filter, and $\top \notin (\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow}$, so $(\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow}$ is an ideal.
By Theorem 6.1.13 $X \subseteq p$ for some prime filter p such that $p \cap (\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow} = \varnothing$. It is easy to check that $z \in X$, so by Definition 6.3.1 $p \in z^\bullet$. Since $\mathcal{Y}$ covers $z^\bullet$, there must exist some $y^\bullet \in \mathcal{Y}$ with $p \in y^\bullet$, that is, $y \in p$. But our assumption that $p \cap (\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow} = \varnothing$ implies that $p \cap \mathcal{Y}^{\bullet\text{-}1} = \varnothing$, contradicting that $y \in p$ and (by Lemma 7.3.6) $y \in \mathcal{Y}^{\bullet\text{-}1}$.

□

Remark 7.3.9. There is something a little odd about the proof of Proposition 7.3.8.
Why did we define X using $\text{-}_{\wedge\uparrow}$ but introduce $\text{-}_{\vee\downarrow}$ only half-way through the proof?
The key detail occurs when we verify that $\text{И}b.(b\ a)\cdot x \in X$ implies $\forall a.x \in X$ (case 4 in the proof above). There, we use the fact that $b\#\mathcal{Y}$ implies $b\#y$ for every y with $y^\bullet \in \mathcal{Y}$. This only follows because $\mathcal{Y}$ (and therefore $\mathcal{Y}^{\bullet\text{-}1}$) has strict finite support.
Now $(\mathcal{Y}^{\bullet\text{-}1})_{\vee\downarrow}$ does not have strict finite support in general, since $y' \leq y$ does not imply $supp(y') \subseteq supp(y)$ in general. Thus we delay its introduction until the second half of the proof, after we have picked b.
Remark 7.2.3 discusses why we introduced strict finite support in the first place. So here in the proof of Proposition 7.3.8, we just have to be a little cautious to only lose the strict finite support property of $\mathcal{Y}^{\bullet\text{-}1}$ *after* we have constructed X.
A converse to Proposition 7.3.8 also holds:

Proposition 7.3.10. *If U is open and compact in $F(\mathcal{D})$ then $U = x^\bullet$ for some unique $x \in |\mathcal{D}|$.*

Proof. By construction in Definition 7.2.1 the open sets of $F(\mathcal{D})$ are unions of strictly finitely-supported sets of sets the form $x^\bullet$. We assumed that U is compact so it has a finite subcover $x_1^\bullet \cup \cdots \cup x_n^\bullet$. We use part 4 of Lemma 6.4.2. Uniqueness is Corollary 6.3.3. □

7.4. Coherent spaces: closure under σ, $\cap$ and $\bigcap^{\#a}$

Coherence usually means that the compact open sets are closed under lattice operations and generate all open sets via sets unions. Our lattices have more structure (notably: a σ-action and $\bigcap^{\#a}$). Also, our notion of 'generating' open sets has some nominal aspects to it.
Definition 7.4.1 is how we extend the notion of coherence to account for this structure. Proposition 7.4.3 then checks that F from Definition 7.2.1 does indeed generate coherent spaces.

Definition 7.4.1. Call a nominal σ-topological space $\mathcal{T}$ **coherent** when:

(1) If U is open and compact then so is $U[a\mapsto u]$ for every $u \in |\mathcal{T}^{\partial}|$.
(2) $|\mathcal{T}|$ is (open and) compact.
(3) If U and V are open and compact then so is $U \cap V$.
(4) If U is open and compact then so is $\bigcap^{\#a} U$.
(5) Every open $U \in opens(\mathcal{T})$ is equal to $\bigcup \mathcal{U}$ for some strictly finitely supported $\mathcal{U} \subseteq cpct(\mathcal{T})$.

Remark 7.4.2. We rewrite Definition 7.4.1 in less precise, but more intuitive language:

(1) Compactness is closed under the σ-action.
(2) Compactness is closed under sets intersection.
(3) Compactness is closed under universal quantification.
(4) Compact open sets are a strictly finitely supported (Subsection 2.4.2) basis for all open sets.

Proposition 7.4.3. *Suppose $\mathcal{D} \in \mathsf{nDi\forall}$. Then $F(\mathcal{D})$ (Definition 7.2.1) is coherent.*

Proof. By Propositions 7.3.8 and 7.3.10 we can identify the compact open sets of $F(\mathcal{D})$ with sets of the form $z^{\bullet}$ for $z \in |\mathcal{D}|$. We now reason as follows:

(1) By Lemma 6.4.1 $x^{\bullet}[a\mapsto u] = (x[a\mapsto u])^{\bullet}$.
(2) By part 1 of Proposition 6.1.15 $\top^{\bullet} = |F(\mathcal{D})|$ (every point contains $\top$).
(3) By part 2 of Lemma 6.4.2 $x^{\bullet} \cap y^{\bullet} = (x \wedge y)^{\bullet}$.
(4) By part 3 of Lemma 6.4.2 $(\forall a.x)^{\bullet} = \bigcap^{\#a} x^{\bullet}$.
(5) By construction in Definition 7.2.1 open sets are strictly finitely supported unions of the $x^{\bullet}$.

□

7.5. Impredicativity

We saw in Subsection 4.5 and Definition 4.5.1 a notion of *impredicativity*, based on the idea that the things we substitute for should map to the things we substitute in. In the context of a topological space, this means that $\mathcal{T}^{\partial}$ should map to open sets. This is Definition 7.5.2, and Theorem 7.6.2 shows how $F(\mathcal{D})$ inherits any impredicative structure of $\mathcal{D}$.

We note an easy corollary of Definitions 7.1.1 and 7.4.1:

Lemma 7.5.1. *Suppose $\mathcal{T}$ is a nominal σ-topological space. Then $opens(\mathcal{T})$ inherits a σ-algebra structure over $\mathcal{T}^{\partial}$.*

If $\mathcal{T}$ is coherent, then also $cpct(\mathcal{T})$ inherits a σ-algebra structure over $\mathcal{T}^{\partial}$.

Proof. This is condition 1 of Definition 7.1.1 and condition 1 of Definition 7.4.1 (if U is open/compact then so is $U[a\mapsto u]$), combined with Proposition 3.4.9. □

We can think of Definition 7.5.2 as a dual to Definition 4.5.1, for the nominal σ-topological spaces from Definition 7.1.1:

Definition 7.5.2. An **impredicative nominal σ-topological space** is a pair $(\mathcal{T}, \partial_{\mathcal{T}})$ where:

— $\mathcal{T}$ is a nominal σ-topological space (Definition 7.1.1).
— $\partial_{\mathcal{T}} : \mathcal{T}^{\partial} \to cpct(\mathcal{T})$ is a morphism of σ-algebras (Definition 4.4.4).

Notation 7.5.3. Following Notation 4.5.3 we introduce some notation for Definition 7.5.2:

— We may drop subscripts and write ∂u for $\partial_{\mathcal{T}} u$ where $u \in |\mathcal{T}^{\partial}|$.
— We may write ∂a for $\partial_{\mathcal{T}}(a_{\mathcal{T}^{\partial}})$ where $a_{\mathcal{T}^{\partial}} = \mathsf{atm}_{\mathcal{D}^{\partial}}(a)$ (Definition 3.1.1).
— We may write $\partial\mathcal{T}$ for $\{\partial u \mid u \in |\mathcal{T}^{\partial}|\} \subseteq cpct(\mathcal{T})$ and call this set the **programs** of $\mathcal{T}$.

The exposition in and following Notation 4.5.3 is also valid here, so we do not repeat it.

THEOREM 7.5.4. *If $\mathcal{D} \in \mathsf{inDi}\forall$ (so $\mathcal{D}$ is impredicative) then $F(\mathcal{D})$ is also naturally impredicative.*

Proof. We take $F(\mathcal{D})^\partial = \mathcal{D}^\partial$ and $\partial_{F(\mathcal{D})}u = (\partial_{\mathcal{D}}u)^\bullet$. (In fact, this is an injection by Theorem 6.4.4.) $\square$

7.6. The map G from coherent spaces to distributive lattices

DEFINITION 7.6.1. Suppose $\mathcal{T}$ is a coherent (Definition 7.4.1) nominal σ-topological space.

Define $G(\mathcal{T})$ as follows:

- $|G(\mathcal{T})| = cpct(\mathcal{T})$ (compact opens) and $G(\mathcal{T})^\partial = \mathcal{T}^\partial$.
- Given $U \in |G(\mathcal{T})|$ and $u \in |G(\mathcal{T})^\partial| = |\mathcal{T}^\partial|$ define $\pi\cdot U = \{\pi\cdot p \mid p \in U\}$ and $U[a\mapsto u] = \{p \mid p[u\leftarrow a] \in U\}$, following Definition 3.4.1.
- $\top$, $\wedge$, $\bot$, $\vee$, and $\forall$ are interpreted as the whole underlying set $\mathcal{T}$, set intersection $\cap$, the empty set $\varnothing$, set union $\cup$, and $\bigcap^{\#a}$ from Definition 5.2.1.

THEOREM 7.6.2. *Continuing Definition 7.6.1, if $\mathcal{T}$ is coherent then $G(\mathcal{T})$ is a nominal distributive lattice with $\forall$.*

Furthermore, if $\mathcal{T}$ is impredicative then so naturally is $G(\mathcal{T})$.

Proof. By Proposition 7.3.2 $\bot$, $\top$, $\vee$, and the permutation action give results in $|G(\mathcal{T})|$. By our assumption that $\mathcal{T}$ is coherent, so do $\wedge$, the σ-action and $\forall$. We use Theorem 5.2.10.

Now suppose $\mathcal{T}$ is impredicative, so it is equipped with a σ-algebra morphism $\partial_{\mathcal{T}} : \mathcal{T}^\partial \to cpct(\mathcal{T})$ (Definition 4.4.4). We take $\partial_{G(\mathcal{T})} = \partial_{\mathcal{T}}$. $\square$

PROPOSITION 7.6.3. *If $\mathcal{D} \in \mathsf{nDi}\forall/\mathsf{inDi}\forall$ then $GF(\mathcal{D})$ is equal to $\mathcal{D}^\bullet$ from Definition 6.4.3, and the map $x \mapsto x^\bullet$ is an isomorphism in $\mathsf{nDi}\forall/\mathsf{inDi}\forall$.*

Proof. By Propositions 7.3.8 and 7.3.10 $|GF(\mathcal{D})| = |\mathcal{D}^\bullet|$. We use Theorem 6.4.4. $\square$

7.7. Sober spaces

DEFINITION 7.7.1. Suppose $\mathcal{T}$ is a nominal σ-topological space and suppose $\mathcal{U} \subseteq opens(\mathcal{T})$ is a filter (Definition 6.1.1) of open sets in $\mathcal{T}$.

Call the filter $\mathcal{U} \subseteq opens(\mathcal{T})$ **completely prime** when for every strictly finitely supported set of open sets $\mathcal{V} \subseteq opens(\mathcal{T})$ if $\bigcup \mathcal{V} \in \mathcal{U}$ then $V \in \mathcal{U}$ for some $V \in \mathcal{V}$.[25]

DEFINITION 7.7.2. Call a nominal σ-topological space $\mathcal{T}$ **sober** when if $\mathcal{U} \subseteq opens(\mathcal{T})$ is completely prime then there exists a unique $t_{\mathcal{U}} \in |\mathcal{T}|$ such that

$$\forall U{\in}opens(\mathcal{T}).U \in \mathcal{U} \Leftrightarrow t_{\mathcal{U}} \in U.$$

Definition 7.7.2 is the standard notion of sobriety, and states intuitively that completely prime filters characterise the underlying points of the space. For the case of coherent spaces, a slightly more economical characterisation is possible and will be useful:

LEMMA 7.7.3. *If $\mathcal{T}$ is coherent then the completely prime filters of open sets are in a natural bijection with the filters of compact open sets, with the bijection given by:*

- *A completely prime filter $\mathcal{U}' \subseteq opens(\mathcal{T})$ corresponds to $\mathcal{U} = \mathcal{U}' \cap cpct(\mathcal{T})$.*
- *A prime filter $\mathcal{U} \subseteq cpct(\mathcal{T})$ corresponds to $\mathcal{U}' = \{U'{\in}opens(\mathcal{T}) \mid U \subseteq U'\}$, the **up-closure** of $\mathcal{U}$ in $opens(\mathcal{T})$.*

Proof. Suppose $\mathcal{U}'$ is a completely prime filter in $opens(\mathcal{T})$. By part 2 of Proposition 7.3.2 compactness is closed under finite unions and it follows that $\mathcal{U}$ is a prime filter in $cpct(\mathcal{T})$.

[25] A *prime* filter satisfies this property—for *finite* $\mathcal{V}$.

Conversely suppose $\mathcal{U} \subseteq cpct(\mathcal{T})$ is a prime filter. We will show that $\mathcal{U}'$ is completely prime.

Suppose $\mathcal{V} \subseteq opens(\mathcal{T})$ is strictly finitely supported and suppose $\bigcup \mathcal{V} \in \mathcal{U}'$, so that $U \subseteq \bigcup \mathcal{V}$ for some $U \in \mathcal{U}$. But this just states that $\mathcal{V}$ covers U, and by compactness $\mathcal{V}$ has a finite subcover $\{V_1, \dots, V_n\} \subseteq \mathcal{V}$. It follows that $\bigcup \{V_1 \cap U, \dots, V_n \cap U\} = U \in \mathcal{U}$. Since $\mathcal{U}$ is prime it follows that $V_i \cap U \in \mathcal{U}$ for some i, and therefore that $V_i \in \mathcal{U}'$.

It is routine to verify that since $\mathcal{T}$ is coherent, the correspondences between $\mathcal{U}$ and $\mathcal{U}'$ defined above are bijective. □

Definition 7.7.4. Suppose $\mathcal{T} \in \sigma\mathsf{Top}$ and $t \in |\mathcal{T}|$. Define

$$t^* = \{U \in cpct(\mathcal{T}) \mid t \in U\},$$

so that $U \in t^* \Leftrightarrow t \in U$.

Recall from Definition 7.2.1 that if $\mathcal{D} \in \mathsf{inDi\forall}$ then $|F(\mathcal{D})|$ is the set of prime filters in $\mathcal{D}$.

Proposition 7.7.5. *If $\mathcal{T} \in \sigma\mathsf{Top}$ is coherent (Definition 7.4.1) then t^* is a prime filter in $G(\mathcal{T})$, and so it is an element of $|FG(\mathcal{T})|$.*

Proof. Conditions 1, 2, and 3 of Definition 6.1.1 are very easy to check.

For condition 4 it suffices to show that if U is open and compact and $\text{И}b.t \in (b\ a)\cdot U$ then $t \in \bigcap^{\#a} U$. This follows by line 4 of Proposition 5.2.8.

It is a fact that t^* is prime, since if $t \in U \cup V$ then $t \in U$ or $t \in V$. □

Corollary 7.7.6. *Suppose $\mathcal{T} \in \sigma\mathsf{Top}$ is coherent and sober. Then $\mathcal{U}$ is a prime filter in $G(\mathcal{T})$ if and only if $\mathcal{U} = t^*$ for some $t \in |\mathcal{T}|$, and that t is unique.*

As a corollary, the map $t \mapsto t^*$ is a bijection between $|\mathcal{T}|$ and $|FG(\mathcal{T})|$.

Proof. By Proposition 7.7.5 t^* is a prime filter in $G(\mathcal{T})$. Conversely if $\mathcal{U}$ is a prime filter in $G(\mathcal{T})$ then since $\mathcal{T}$ is sober we can use the correspondence of Lemma 7.7.3 to exhibit $\mathcal{U}$ as $t^*_{\mathcal{U}'}$ where $\mathcal{U}' = \{U' \mid \exists U{\in}\mathcal{U}.U \subseteq U'\}$, and this $t_{\mathcal{U}'}$ is unique. □

Lemma 7.7.7. *If $\mathcal{T} \in \sigma\mathsf{Top}$ is coherent then $\mathcal{T}$ is sober if and only if the map $t \in |\mathcal{T}| \longmapsto t^* \in FG(\mathcal{T})$ (a point maps to the prime filter of compact open sets containing it) is a bijection.*

Proof. We just combine Definitions 7.7.2 and 7.7.4 with the correspondence of Lemma 7.7.3. □

Recall the definitions of $x^\bullet$, p^*, and $\mathcal{U}^{\bullet\text{-}1}$ from Definitions 6.3.1, 7.7.4, and 7.3.5 respectively.

Corollary 7.7.8. *Suppose $\mathcal{D} \in \mathsf{nDi\forall}$. Then:*

(1) *If $p \in |F(\mathcal{D})|$ then $(p^*)^{\bullet\text{-}1} = p$.*
(2) *If $\mathcal{U} \in |FGF(\mathcal{D})|$ then $(\mathcal{U}^{\bullet\text{-}1})^* = \mathcal{U}$.*

As a corollary, $F(\mathcal{D})$ is sober.

Proof. We just unravel definitions and see that:

(1) $x \in p$ if and only if $p \in x^\bullet$ if and only if $x^\bullet \in p^*$ if and only if $x \in (p^*)^{\bullet\text{-}1}$.
(2) $x^\bullet \in \mathcal{U}$ if and only if (by Lemma 7.3.6) $x \in \mathcal{U}^{\bullet\text{-}1}$ if and only if $\mathcal{U}^{\bullet\text{-}1} \in x^\bullet$ if and only if $x^\bullet \in (\mathcal{U}^{\bullet\text{-}1})^*$.

The corollary follows from Lemma 7.7.7. □

7.8. Nominal spectral spaces

Definition 7.8.1. Call an impredicative nominal σ-topological space $\mathcal{T}$ (Definition 7.5.2) **spectral** when it is coherent (Definition 7.4.1) and sober (Definition 7.7.2).
We call an impredicative coherent sober nominal σ-topological space a **nominal spectral space**, for short.

Proposition 7.8.2. *If $\mathcal{D} \in \mathsf{nDi}\forall$ then $F(\mathcal{D})$ (Definition 7.2.1) is spectral. If $\mathcal{D} \in \mathsf{inDi}\forall$ then $F(\mathcal{D})$ is impredicative and spectral, that is, $F(\mathcal{D})$ is a nominal spectral space.*

Proof. By Theorem 7.2.2, Proposition 7.4.3, and Corollary 7.7.8, and by Theorem 7.5.4 for impredicativity. $\square$

8. MORPHISMS OF NOMINAL SPECTRAL SPACES

8.1. The definition of $\mathsf{inSpect}\forall$, and F viewed as a functor to it

We see from Definition 7.6.1 that we obtain a nominal distributive lattice with $\forall$ from an impredicative nominal spectral space by taking the lattice of compact open sets.

A *spectral* morphism usually taken to be a map of points whose inverse preserves the property of being compact. Our compact sets have permutation and σ-actions (and our points have permutation and ꝺ-actions) so we need morphisms to interact appropriately with this extra structure. This is Definition 8.1.1.

Then, we extend F from Definition 7.2.1 to act on morphisms, and check that this does indeed yield a functor. This is Definition 8.1.5 and Proposition 8.1.8. Theorem 8.1.9 packages this all up into a theorem.

Definition 8.1.1. Suppose $\mathcal{S}$ and $\mathcal{T}$ are nominal spectral spaces (Definition 7.8.1).
Suppose $g \in |\mathcal{T}| \to |\mathcal{S}|$. Then:

— Call g **continuous** when $X \in opens(\mathcal{S})$ implies $g^{-1}(X) \in opens(\mathcal{T})$ (inverse image of an open is open).
— Call g **spectral** when $X \in cpct(\mathcal{S})$ implies $g^{-1}(X) \in cpct(\mathcal{T})$ (inverse image of a compact open is compact open).

Call $g = (g_{\mathcal{S}}, g^{\partial}_{\mathcal{S}})$ a **morphism** from $\mathcal{T}$ to $\mathcal{S}$, and (dropping subscripts) write $g = (g, g^{\partial}) : \mathcal{T} \to \mathcal{S}$ when:

(1) g is equivariant, meaning that $\pi \cdot g(t) = g(\pi \cdot t)$.
(2) $g \in |\mathcal{T}| \to |\mathcal{S}|$ is continuous and spectral, and g^{∂} is a σ-algebra morphism (Definition 4.4.4) from $\mathcal{S}^{\partial}$ to $\mathcal{T}^{\partial}$.
So g goes from $\mathcal{T}$ to $\mathcal{S}$ but g^{∂} goes from $\mathcal{S}^{\partial}$ to $\mathcal{T}^{\partial}$.
(3) g^{-1} maps atoms-as-programs to themselves, meaning that $g^{-1}(\partial_{\mathcal{S}} a) = \partial_{\mathcal{T}} a$ for every atom a.
(4) g^{-1} commutes with the ꝺ-action, meaning that for $u \in |\mathcal{S}^{\partial}|$ and $p \in |\mathcal{T}|$

$$g(p)[u \hookleftarrow a] = g(p[g^{\partial}(u) \hookleftarrow a]).$$

(See Lemma 8.1.3 for another view of this condition.)

Write $\mathsf{inSpect}\forall$ for the category of nominal spectral spaces, and morphisms between them.

Lemmas 8.1.2 and 8.1.3 are direct derivatives of condition 4 of Definition 8.1.1:

Lemma 8.1.2. *If $g : \mathcal{T} \to \mathcal{S}$ then for $p \in |\mathcal{T}|$ and $n \in \mathbb{A}$, $g(p)[n_{\mathcal{S}^{\partial}} \hookleftarrow a] = g(p[n_{\mathcal{T}^{\partial}} \hookleftarrow a])$.*

Proof. The proof is just a special case of condition 4 of Definition 8.1.1, noting that g^{∂} is assumed to be a σ-algebra morphism, and by condition 1 of Definition 4.4.4 $g^{\partial}(n_{\mathcal{S}^{\partial}}) = n_{\mathcal{T}^{\partial}}$. $\square$

LEMMA 8.1.3. *Suppose $g : \mathcal{T} \to \mathcal{S}$ and suppose $U \in cpct(\mathcal{S})$ and $u \in |\mathcal{S}^{\partial}|$. Then*

$$g^{-1}(U[a\mapsto u]) = g^{-1}(U)[a\mapsto g^{\partial}(u)].$$

(The σ-action $U[a\mapsto u]$ and $U[a\mapsto g^{\partial}(u)]$ is from Definition 3.4.1.)

Proof. Suppose $p \in |\mathcal{T}|$. Then:

$$\begin{array}{rlr}
p \in g^{-1}(U[a\mapsto u]) & \Leftrightarrow g(p) \in U[a\mapsto u] & \text{Pointwise action} \\
& \Leftrightarrow \text{И}c.g(p)[u\leftarrow c] \in (c\ a)\cdot U & \text{Proposition 3.4.2} \\
& \Leftrightarrow \text{И}c.g(p[g^{\partial}(u)\leftarrow c]) \in (c\ a)\cdot U & \text{C4 of Def 8.1.1} \\
& \Leftrightarrow \text{И}c.p[g^{\partial}(u)\leftarrow c] \in (c\ a)\cdot g^{-1}(U) & \text{Pointwise action} \\
& \Leftrightarrow p \in g^{-1}(U)[a\mapsto g^{\partial}(u)] & \text{Proposition 3.4.2}
\end{array}$$

□

We take a moment to note an interaction of Lemma 8.1.3 with condition 3 of Definition 8.1.1:

COROLLARY 8.1.4. *g^{-1} maps programs to programs, meaning that $g^{-1}(\partial_{\mathcal{S}} u) = \partial_{\mathcal{T}} u$ for every $u \in |\mathcal{S}^{\partial}|$.*

Proof. By condition 3 of Definition 8.1.1 $g^{-1}(\partial_{\mathcal{S}} a) = \partial_{\mathcal{T}} a$. We apply $[a\mapsto u]$ to both sides and use Lemma 8.1.3 to deduce that $g^{-1}(\partial_{\mathcal{S}} u) = \partial_{\mathcal{T}} u$. □

Definition 8.1.5 extends Definition 7.2.1 from objects to morphisms:

DEFINITION 8.1.5. Given a morphism $f = (f_{\mathcal{D}}, f^{\partial}_{\mathcal{D}}) : \mathcal{D} \to \mathcal{D}'$ in inDi∀ (Definitions 4.5.8, 4.4.5, and 4.4.4) define $F(f) : F(\mathcal{D}') \to F(\mathcal{D})$ by

— $F(f)_{\mathcal{D}'}(p') = f^{-1}_{\mathcal{D}}(p')$ where $p' \in |F(\mathcal{D}')|$ (so p' is a point—a prime filter—in $\mathcal{D}'$) and
— $F(f)^{\partial}_{\mathcal{D}}(u) = f_{\mathcal{D}^{\partial}}(u)$ where $u \in |\mathcal{D}^{\partial}|$.

That is, without subscripts:

$$\boxed{x \in F(f)(p) \Leftrightarrow f(x) \in p \qquad \text{and} \qquad F(f)^{\partial}(u) = f^{\partial}(u)}$$

We now work towards proving that $F(f)$ maps points to points in Proposition 8.1.8.

LEMMA 8.1.6. *Suppose $f = (f_{\mathcal{D}}, f^{\partial}_{\mathcal{D}}) : \mathcal{D} \to \mathcal{D}'$ is a morphism in* inDi∀ *and suppose $x \in |\mathcal{D}|$. Then $F(f)^{-1}(x^{\bullet}) = (f(x))^{\bullet}$.*

Proof. We reason as follows, where $p \in |points(\mathcal{D}')|$:

$$\begin{array}{rlr}
p \in F(f)^{-1}(x^{\bullet}) & \Leftrightarrow F(f)(p) \in x^{\bullet} & \text{Inverse image} \\
& \Leftrightarrow x \in F(f)(p) & \text{Definition 6.3.1} \\
& \Leftrightarrow f(x) \in p & \text{Definition 8.1.5} \\
& \Leftrightarrow p \in (f(x))^{\bullet} & \text{Definition 6.3.1}
\end{array}$$

□

LEMMA 8.1.7. *Suppose $f : \mathcal{D} \to \mathcal{D}'$ is a morphism (Definition 4.5.8). Then if $p \subseteq |\mathcal{D}'|$ is a filter then so is $F(f)(p) = f^{-1}(p)$, and if p is prime then so is $f^{-1}(p)$.*

Proof. Suppose p is a filter. We check the conditions of Definition 6.1.1 for $f^{-1}(p)$, freely using the fact that $x \in f^{-1}(p)$ if and only if $f(x) \in p$:

(1) $\bot \notin f^{-1}(p)$. By assumption in Definition 4.4.5 $f(\bot) = \bot$, and by condition 1 of Definition 6.1.1 $\bot \notin p$.
(2) *$x \in f^{-1}(p)$ and $x \leq y$ implies $y \in f^{-1}(p)$.* Immediate.
(3) *$x \in f^{-1}(p)$ and $y \in f^{-1}(p)$ implies $x \wedge y \in f^{-1}(p)$.* Since $f(x \wedge y) = f(x) \wedge f(y)$.

(4) *If* $\text{И}b.((b\ a)\cdot x \in f^{\text{-}1}(p))$ *then* $\forall a.x \in f^{\text{-}1}(p)$. Suppose $\text{И}b.(b\ a)\cdot x{\in}f^{\text{-}1}(p)$. It follows from Definition 4.4.5 that $\text{И}b.(b\ a)\cdot f(x){\in}p$, so by condition 3 of Definition 6.1.1 $\forall a.f(x) \in p$. By Definition 4.4.5 again $\forall a.f(x) = f(\forall a.x)$. The result follows.

Now suppose p is prime and suppose $x{\vee}y \in f^{\text{-}1}(p)$, so that $f(x{\vee}y) \in p$. From Definition 4.4.5 we have $f(x){\vee}f(y) \in p$. Since p is prime, either $f(x) \in p$ or $f(y) \in p$. □

Proposition 8.1.8. *If $f : \mathcal{D} \to \mathcal{D}'$ is a morphism (Definition 4.5.8) then $F(f)$ from Definition 8.1.5 is a morphism from $F(\mathcal{D}')$ to $F(\mathcal{D})$ (Definition 8.1.1).*

Proof. By Lemma 8.1.7 $F(f)$ maps prime filters of $\mathcal{D}'$ to prime filters of $\mathcal{D}$—that is, $F(f)_{\mathcal{D}'} \in |F(\mathcal{D}')| \to |F(\mathcal{D})|$.

Now we show that $F(f)$ is a morphism. We verify the properties of Definition 8.1.1. By definition $F(f)^{\partial} = f^{\partial}$ which by construction is a σ-algebra morphism from $\mathcal{D}^{\partial}$ to $\mathcal{D}'^{\partial}$. Also:

(1) *$F(f)$ is equivariant.* We briefly sketch the reasoning; in step $(*)$ we use condition 1 of Definition 8.1.1 for f:

$$x \in \pi\cdot(F(f)(p)) \Leftrightarrow \pi^{\text{-}1}\cdot x \in F(f)(p) \Leftrightarrow f(\pi^{\text{-}1}\cdot x) \in p \overset{(*)}{\Leftrightarrow} \pi^{\text{-}1}\cdot f(x) \in p$$
$$\Leftrightarrow f(x) \in \pi\cdot p \Leftrightarrow x \in F(f)(\pi\cdot p)$$

(2) *$F(f)^{\text{-}1}$ is continuous and spectral.* We must prove two things:
— *$F(f)^{\text{-}1}$ maps open sets to open sets.* By construction $F(f)^{\text{-}1}$ preserves unions, and by construction every $X \in \mathit{opens}(F(\mathcal{D}))$ is a strictly finitely supported union $\bigcup_{i\in I} x_i^{\bullet}$ for $x_i \in |\mathcal{D}|$. By Lemma 8.1.6 $F(f)^{\text{-}1}(x_i^{\bullet}) = f(x_i)^{\bullet}$ and by Corollary 6.3.4 and Theorem 2.3.1 $\mathit{supp}(f(x_i)^{\bullet}) \subseteq \mathit{supp}(x_i^{\bullet})$ for every $i{\in}I$. It follows that $F(f)^{\text{-}1}(X) = \bigcup_{i\in I} f(x_i)^{\bullet}$ and this is a strictly finitely supported union and so is open in $F(\mathcal{D}')$.
— *$F(f)^{\text{-}1}$ maps compact sets to compact sets.* By Lemma 8.1.6 $F(f)^{\text{-}1}(x^{\bullet}) = f(x)^{\bullet}$. We use Propositions 7.3.8 and 7.3.10.

(3) *$F(f)^{\text{-}1}$ maps atoms-as-programs to themselves.* By Lemma 8.1.6 $F(f)^{\text{-}1}((\partial_{\mathcal{D}} a)^{\bullet}) = (f(\partial_{\mathcal{D}} a))^{\bullet}$ and by assumption in Definition 4.5.8 $f(\partial_{\mathcal{D}} a) = \partial_{\mathcal{D}'} a$.

(4) *$F(f)$ commutes with the ɒ-action.* Suppose $p' \in |F(\mathcal{D}')|$ and $u \in |F(\mathcal{D})^{\partial}|$ and $x \in |\mathcal{D}|$. Following Theorem 7.5.4 $|F(\mathcal{D})^{\partial}| = |\mathcal{D}^{\partial}|$ so $u \in |\mathcal{D}^{\partial}|$. We reason as follows:

$$\begin{array}{rll} x \in F(f)(p'[f^{\partial}(u){\hookleftarrow}a]) & \Leftrightarrow f(x) \in p'[f^{\partial}(u){\hookleftarrow}a] & \text{Definition 8.1.5} \\ & \Leftrightarrow f(x)[a{\mapsto}f^{\partial}(u)] \in p' & \text{Proposition 3.3.2} \\ & \Leftrightarrow f(x[a{\mapsto}u]) \in p' & \text{C3 of Def 4.4.4} \\ & \Leftrightarrow x[a{\mapsto}u] \in F(f)(p') & \text{Definition 8.1.5} \\ & \Leftrightarrow x \in (F(f)(p'))[u{\hookleftarrow}a] & \text{Proposition 3.3.2} \end{array}$$

□

Theorem 8.1.9. *F from Definitions 7.2.1 and 8.1.5 is a functor from* $\mathsf{inDi}\forall$ *to* $\mathsf{inSpect}\forall^{op}$.

Proof. By Theorem 7.2.2 $F(\mathcal{D})$ is a σ-topological space. By Theorem 7.5.4 $F(\mathcal{D})$ is compact and impredicative. By Proposition 7.4.3 $F(\mathcal{D})$ is coherent. By Corollary 7.7.8 $F(\mathcal{D})$ is sober.

Furthermore if $f : \mathcal{D} \to \mathcal{D}'$ in $\mathsf{inDi}\forall$ then by Proposition 8.1.8 $F(f)$ is a morphism from $F(\mathcal{D}')$ to $F(\mathcal{D})$. The result follows by some easy calculations. □

8.2. The action of G on morphisms in $\mathsf{inSpect}\forall$

In Subsection 8.1 we went from $\mathsf{inDi}\forall$ to $\mathsf{inSpect}\forall$. Now we go back.

So Definition 8.2.1 mapping $\mathsf{inSpect}\forall$ to $\mathsf{inDi}\forall$ is the dual to Definition 8.1.5 mapping $\mathsf{inDi}\forall$ to $\mathsf{inSpect}\forall$:

Definition 8.2.1. Given $g : \mathcal{T}' \to \mathcal{T}$ in $\mathsf{inSpect}\forall$ define $G(g) : G(\mathcal{T}) \to G(\mathcal{T}')$ by $G(g)_{\mathcal{T}}(U) = g^{-1}_{\mathcal{T}'}(U)$ and $G(g)_{\mathcal{T}^\partial}(u) = u$, that is (without subscripts):

$$\boxed{t' \in G(g)(U) \Leftrightarrow g(t') \in U \quad \text{and} \quad G(g)^\partial(u) = g^\partial(u)}$$

Proposition 8.2.2. *G from Definitions 7.6.1 and 8.2.1 is a functor from $\mathsf{inSpect}\forall^{op}$ to $\mathsf{inDi}\forall$.*

Proof. By Theorem 7.6.2 G maps objects of $\mathsf{inSpect}\forall^{op}$ to objects of $\mathsf{inDi}\forall$.

Now consider a morphism $g : \mathcal{T}' \to \mathcal{T}$ in the sense of Definition 8.1.1; the interesting part is to check that $G(g)_{\mathcal{T}}$—that is, $g^{-1}_{\mathcal{T}'}$—is a morphism in the sense of Definition 4.5.8.

We may drop subscripts henceforth. If $U \in cpct(\mathcal{T})$, so U is compact, and since g is assumed spectral in Definition 8.1.1 we know $g^{-1}(U)$ is also compact. It is routine to check that g^{-1} preserves the top $\top$ and bottom elements ($|\mathcal{T}|$ and $\varnothing$ respectively) and interacts correctly with intersections and unions.

It remains to show that g^{-1} is equivariant, commutes with the σ-action, and commutes with $\bigcap^{\#a}$.

Equivariance, meaning that $\pi{\cdot}g^{-1}(U) = g^{-1}(\pi{\cdot}U)$, is immediate by Theorem 2.3.1 (a proof by concrete calculations using Proposition 3.4.2 and condition 1 of Definition 8.1.1 is also possible). Commuting with the σ-action, meaning that $g^{-1}_{\mathcal{T}'}(U)[a{\mapsto}g^\partial_{\mathcal{T}}(v)] = g^{-1}_{\mathcal{T}'}(U[a{\mapsto}v])$, is Lemma 8.1.3.

Finally we check that g^{-1} commutes with $\bigcap^{\#a}$:

$$\begin{array}{r@{\ }l@{\qquad}l}
t \in g^{-1}(\bigcap^{\#a} U) & \Leftrightarrow g(t) \in \bigcap^{\#a} U & \text{Pointwise action} \\
& \Leftrightarrow \forall n{\in}\mathbb{A}.g(t) \in U[a{\mapsto}n] & \text{Line 2 of Prop 5.2.8} \\
& \Leftrightarrow \forall n{\in}\mathbb{A}.\text{И}c.g(t)[n{\leftarrow}c] \in (c\ a){\cdot}U & \text{Proposition 3.4.2} \\
& \Leftrightarrow \forall n{\in}\mathbb{A}.\text{И}c.g(t[n{\leftarrow}c]) \in (c\ a){\cdot}U & \text{Lemma 8.1.2} \\
& \Leftrightarrow \forall n{\in}\mathbb{A}.\text{И}c.t[n{\leftarrow}c] \in (c\ a){\cdot}g^{-1}(U) & \text{Pointwise action, Thm 2.3.1} \\
& \Leftrightarrow \forall n{\in}\mathbb{A}.t \in g^{-1}(U)[a{\mapsto}n] & \text{Proposition 3.4.2} \\
& \Leftrightarrow t \in \bigcap^{\#a} g^{-1}(U) & \text{Line 2 of Prop 5.2.8}
\end{array}$$

Thus $G(g)$ is a morphism in $\mathsf{inDi}\forall$. □

Notice of the last case above that we prove a property of an infinite intersection using its characterisation as a universal atoms-quantification; we can do this thanks to Proposition 5.2.8, and it makes the proof much easier.

8.3. The equivalence

In Subsections 8.1 and 8.2 we considered two functors $F : \mathsf{inDi}\forall \longrightarrow \mathsf{inSpect}\forall$ and $G : \mathsf{inSpect}\forall \longrightarrow \mathsf{inDi}\forall$. They are dual; the key is to observe that $FG(\mathcal{T})$ is isomorphic to $\mathcal{T}$. This is Lemma 8.3.1 and Proposition 8.3.2. Theorem 8.3.3 puts it all together.

Lemma 8.3.1. *Suppose $\mathcal{T} \in \mathsf{inSpect}\forall$ and $t \in |\mathcal{T}|$. If we give $t^* \in |FG(\mathcal{T})|$ from Definition 7.7.4 the pointwise ɤ-action from Definition 3.3.1 then for $U \in cpct(\mathcal{T})$*

$$U \in t^*[u{\leftarrow}a] \Leftrightarrow \text{И}c.(c\ a){\cdot}U \in (t[u{\leftarrow}c])^*.$$

Proof. We reason as follows:

$$\begin{array}{r@{\ }l@{\qquad}l}
U \in t^*[u{\leftarrow}a] & \Leftrightarrow U[a{\mapsto}u] \in t^* & \text{Proposition 3.3.2} \\
& \Leftrightarrow t \in U[a{\mapsto}u] & \text{Definition 7.7.4} \\
& \Leftrightarrow \text{И}c.t[u{\leftarrow}c] \in (c\ a){\cdot}U & \text{Proposition 3.4.2} \\
& \Leftrightarrow \text{И}c.(c\ a){\cdot}U \in (t[u{\leftarrow}c])^* & \text{Definition 7.7.4}
\end{array}$$

□

Recall from Definitions 7.2.1 and 7.6.1 that $FG(\mathcal{T})$ is a topological space whose points are prime filters of compact opens in $\mathcal{T}$.

PROPOSITION 8.3.2. *If* $\mathcal{T} \in \mathsf{inSpect}\forall$ *then* $\alpha_{\mathcal{T}}$ *mapping* $t \in |\mathcal{T}|$ *to* $t^* \in |FG(\mathcal{T})|$ *defines an isomorphism in* $\mathsf{inSpect}\forall$ *between* $\mathcal{T}$ *and* $FG(\mathcal{T})$.

Proof. Injectivity and surjectivity are Corollary 7.7.6. Commutativity with the σ-action is Lemma 8.3.1, as can be checked by unravelling definitions.

We also need to show that α is continuous. The reasoning is standard [BS81, Section 4] so we just sketch it. First, if $U \in G(\mathcal{T})$ (so U is a compact open set of $\mathcal{T}$) consider the inverse image under -^* of $U^\bullet$.[26]

$$\begin{aligned} t \in \alpha^{-1}_{\mathcal{T}}(U^\bullet) &\Leftrightarrow \alpha_{\mathcal{T}}(t) \in U^\bullet \\ &\Leftrightarrow \{U' \mid t \in U'\} \in U^\bullet \\ &\Leftrightarrow U \in \{U' \mid t \in U'\} \\ &\Leftrightarrow t \in U \end{aligned}$$

Thus, $\alpha^{-1}_{\mathcal{T}}(U^\bullet) = U$.

Now by construction any open set in $FG(\mathcal{T})$ is a union of $U^\bullet$, and it is a fact that the inverse image function $\alpha^{-1}_{\mathcal{T}}$ preserves these unions. It follows that the inverse image of an open set is open. $\square$

THEOREM 8.3.3. $G : \mathsf{inSpect}\forall^{\mathsf{op}} \to \mathsf{inDi}\forall$ *defines an equivalence between* $\mathsf{inDi}\forall$ *and* $\mathsf{inSpect}\forall^{op}$.

Proof. We use [Mac71, Theorem 1, Chapter IV, Section 4].

- *G is essentially surjective on objects.* This is Proposition 7.6.3.
- *G is faithful.* Suppose $g_1, g_2 : \mathcal{T} \to \mathcal{S} \in \mathsf{inSpect}\forall$ and $g_1 \neq g_2$; the interesting case here is then that there exists $p \in |\mathcal{T}|$ such that $g_1(p) \neq g_2(p)$. (F and G leave programs unchanged, so we can elide g_1^∂ and g_2^∂.)
  By assumption $\mathcal{S}$ is coherent and sober, so that by Lemma 7.7.7 $g_1(p)^*$ and $g_2(p)^*$—these are the sets of compact open sets in $\mathcal{S}$ containing $g_1(p)$ and $g_2(p)$ respectively—are not equal.
  Thus there exists a compact open set $U \in opens(\mathcal{S})$ with $g_1(p) \in U$ and $g_2(p) \notin U$. Examining Definition 8.2.1 we see that $p \in G(g_1)(U)$ and $p \notin G(g_2)(U)$. Thus, $G(g_1) \neq G(g_2)$.
- *G is full.* Given $\mathcal{S}, \mathcal{T}$ in $\mathsf{inSpect}\forall$ and $f : G(\mathcal{S}) \to G(\mathcal{T})$ in $\mathsf{inDi}\forall$ we construct a morphism $g : \mathcal{T} \to \mathcal{S}$ in $\mathsf{inSpect}\forall$ such that $G(g) = f$.
  By Proposition 8.3.2 $\alpha_{\mathcal{T}} : \mathcal{T} \to FG(\mathcal{T})$ mapping t to t^* is an isomorphism in $\mathsf{inSpect}\forall$. Set $g = \alpha^{-1}_{\mathcal{S}} \circ F(f) \circ \alpha_{\mathcal{T}}$. By routine calculations we can check that $G(g)(U') = f(U')$ for every $U' \in |G(\mathcal{S})|$.

$\square$

III. ADDING APPLICATION AND ITS TOPOLOGICAL DUAL THE COMBINATION OPERATOR

So far we have seen $\mathsf{inDi}\forall$ and $\mathsf{inSpect}\forall$, and Theorem 8.3.3 is a topological duality theorem relating them. This is in itself an interesting result: duality for an impredicative propositional logic (propositional logic with quantifiers over propositions; the reader might be familiar with this kind of logical system in the form of the type system of System F [GTL89]).

However, to model the λ-calculus we need more structure.

This is developed in Section 9, and our results so far are extended accordingly—culminating in Subsection 9.6 with Theorem 9.6.6.

$$
\begin{array}{ll}
(\bullet\epsilon) & (u\multimap\!\!\bullet\, x)\bullet u \leq x \\
(\bullet\eta) & x \leq u\multimap\!\!\bullet(x\bullet u)
\end{array}
$$

Fig. 2: Adjointness properties for $\bullet$ and $\multimap\!\!\bullet$

$$
\begin{array}{llll}
(\sigma\bullet) & & (x\bullet y)[b\mapsto v] = (x[b\mapsto v])\bullet(y[b\mapsto v]) & \\
(\sigma\multimap\!\!\bullet) & & (\partial b\multimap\!\!\bullet x)[a\mapsto v] = \partial b\multimap\!\!\bullet(x[a\mapsto v]) & \\
(\bullet\bot) & & \bot\bullet u = \bot & x\bullet\bot = \bot \\
(\bullet\land) & & (x\land y)\bullet u \leq (x\bullet u)\land(y\bullet u) & x\bullet(u\land v) \leq (x\bullet u)\land(x\bullet v) \\
(\bullet\lor) & & (x\lor y)\bullet u = (x\bullet u)\lor(y\bullet u) & x\bullet(u\lor v) = (x\bullet u)\lor(x\bullet v) \\
(\bullet\forall) & b\#u \Rightarrow & (\bigwedge^{\#b} x)\bullet u \leq \bigwedge^{\#b}(x\bullet u) & \\
(\multimap\!\!\bullet\land) & & u\multimap\!\!\bullet(x\land y) = (u\multimap\!\!\bullet x)\land(u\multimap\!\!\bullet y) & \\
(\multimap\!\!\bullet\lor) & & u\multimap\!\!\bullet(x\lor y) \geq (u\multimap\!\!\bullet x)\lor(u\multimap\!\!\bullet y) & \\
(\multimap\!\!\bullet\forall) & b\#u \Rightarrow & \bigwedge^{\#b}(u\multimap\!\!\bullet x) \leq u\multimap\!\!\bullet(\bigwedge^{\#b} x) &
\end{array}
$$

Fig. 3: Compatibility properties for $\bullet$ and $\multimap\!\!\bullet$

9. INDI$\forall_\bullet$ AND INSPECT$\forall_\bullet$

9.1. Adding $\bullet$ and $\multimap\!\!\bullet$ to inDi$\forall$ to get inDi$\forall_\bullet$

Definition 9.1.1. We extend the notion of an impredicative nominal distributive lattice with $\forall$ from Definition 4.5.1 with two equivariant operators $\bullet : (\mathcal{X}\times\mathcal{X}) \Rightarrow \mathcal{X}$ and $\multimap\!\!\bullet : (\mathcal{X}\times\mathcal{X}) \Rightarrow \mathcal{X}$, written infix as $x\bullet y$ and $y\multimap\!\!\bullet x$.
They must be **adjoint** as described in Figure 2, and they must be **compatible** as described in Figure 3 (the notation ∂b is from Notation 4.5.3).

Remark 9.1.2. In categorical terminology, axiom $(\bullet\epsilon)$ is a *counit* and $(\bullet\eta)$ is a *unit*. In Proposition 10.2.4 we will derive β-reduction from $(\bullet\epsilon)$ and η-expansion from $(\bullet\eta)$. Later on in Remark 11.3.7 we will examine how $\multimap\!\!\bullet$ behaves in a concrete model.

Remark 9.1.3. In the presence of $(\sigma\#)$ there is redundancy in $(\sigma\multimap\!\!\bullet)$; we could take $(\sigma\multimap\!\!\bullet)$ to be $\partial b\multimap\!\!\bullet(x[a\mapsto v]) \leq (\partial b\multimap\!\!\bullet x)[a\mapsto v]$ and get the reverse inequality from Lemma 9.1.9 (below). The form given in Definition 9.1.1 is slightly more convenient to work with.

Remark 9.1.4. $(\multimap\!\!\bullet\forall)$ is an inequality, not an equality. This seems odd, given that $(\multimap\!\!\bullet\land)$ is an equality; is not $\forall$ intuitively an infinite conjunction or a fresh-finite limit? The reason is that this reflects the *in*equality in Lemma 9.1.9 below. To see how this works, we refer the interested reader to the case of $(\multimap\!\!\bullet\forall)$ in Lemma 9.6.3.[27]

Remark 9.1.5. $(\sigma\multimap\!\!\bullet)$ might seem odd: why $\partial b\multimap\!\!\bullet x$ and not $y\multimap\!\!\bullet x$? The precise technical reason is in Proposition 11.6.4; the $\partial b\multimap\!\!\bullet x$ form is what we prove of our canonical syntactic model $points_\Pi$.
We package our definitions up as a category:

Definition 9.1.6. Continuing Definition 9.1.1, write inDi$\forall_\bullet$ for the category with objects impredicative nominal distributive lattices with $\forall$, $\bullet$, and $\multimap\!\!\bullet$, and morphisms are morphisms in inDi$\forall$

[26]Unpacking Definitions 6.3.1 and 7.6.1, $U^\bullet$ is the set of prime filters of compact opens of $\mathcal{T}$ of which U is an element.
[27]The even more interested reader is referred to Proposition 9.3.7, where $(\multimap\!\!\bullet\forall)$ is just what we need for the final stages of the proof.

(Definition 4.5.9) that commute with $\bullet$ and $\multimap\!\!\bullet$ in the following sense:

$$\begin{aligned} f(x\bullet y) &= f(x)\bullet f(y) \quad \text{and} \\ f(\partial b \multimap\!\!\bullet x) &= \partial b \multimap\!\!\bullet f(x). \end{aligned}$$

For more on ∂b see Remarks 4.5.4 and 4.5.10.

We conclude with technical lemmas concerning the interaction of $\bullet$ and $\multimap\!\!\bullet$ with $\leq$ and the σ-action. For the rest of this subsection we fix $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ and $x, y, x', y' \in |\mathcal{D}|$ and $u \in |\mathcal{D}^\partial|$.

LEMMA 9.1.7.*(1) If $x \leq x'$ then $x\bullet y \leq x'\bullet y$.*
(2) If $x \leq x'$ then $y \multimap\!\!\bullet x \leq y \multimap\!\!\bullet x'$.

Proof. It is a fact that $x \leq x'$ if and only if $x \wedge x' = x$. We reason as follows:

(1) $x\bullet y = (x\wedge x')\bullet y \overset{(\bullet\wedge)}{\leq} (x\bullet y)\wedge(x'\bullet y) \leq x'\bullet y$

(2) $y \multimap\!\!\bullet x = y \multimap\!\!\bullet (x\wedge x') \overset{(\multimap\bullet\vee)}{=} (y \multimap\!\!\bullet x)\wedge(y \multimap\!\!\bullet x') \leq y \multimap\!\!\bullet x'$

∎

LEMMA 9.1.8. *$x\bullet y \leq z$ if and only if $x \leq y \multimap\!\!\bullet z$.*

Proof. The reasoning is standard:

— If $x\bullet y \leq z$ then $x \overset{(\bullet\eta)}{\leq} y \multimap\!\!\bullet (x\bullet y) \overset{\text{L9.1.7(2)}}{\leq} y \multimap\!\!\bullet z$.

— If $x \leq y \multimap\!\!\bullet z$ then $x\bullet y \overset{\text{L9.1.7(1)}}{\leq} (y \multimap\!\!\bullet z)\bullet y \overset{(\bullet\epsilon)}{\leq} z$.

∎

LEMMA 9.1.9. $(y \multimap\!\!\bullet x)[a\mapsto u] \leq y[a\mapsto u] \multimap\!\!\bullet x[a\mapsto u]$.

Proof. We reason as follows:

$$\begin{aligned} (y \multimap\!\!\bullet x)[a\mapsto u] \leq y[a\mapsto u] \multimap\!\!\bullet x[a\mapsto u] &\Leftrightarrow (y \multimap\!\!\bullet x)[a\mapsto u]\bullet y[a\mapsto u] \leq x[a\mapsto u] && \text{Lemma 9.1.8} \\ &\Leftrightarrow ((y \multimap\!\!\bullet x)\bullet y)[a\mapsto u] \leq x[a\mapsto u] && (\sigma\bullet) \\ &\Leftarrow x[a\mapsto u] \leq x[a\mapsto u] && \text{Lemma 4.3.2, } (\bullet\eta) \end{aligned}$$

∎

9.2. The combination operator $\circ$: a topological dual to $\bullet$ and $\multimap\!\!\bullet$

In Subsection 9.1 we extended $\mathsf{inDi}\forall_\bullet$ with extra structure of $\bullet$ and $\multimap\!\!\bullet$.

We can expect this to be reflected in the topologies by some kind of extension with a structure dual to $\bullet$ and $\multimap\!\!\bullet$.[28]

What should dually correspond to $\bullet$ and $\multimap\!\!\bullet$? Remarkably, this requires only a little more structure on points: see the combination operator $\circ$ in Definition 9.2.1.

9.2.1. The basic definition. Recall the finitely supported powerset $pow(\text{-})$ from Subsection 2.4.1.

DEFINITION 9.2.1. We define an **$\mho\circ$-algebra** by extending the notion of $\mho$-algebra $\mathcal{P}$ from Definition 3.2.1 with an equivariant **combination** operator

$$\circ : (\mathcal{P} \times \mathcal{P}) \Rightarrow pow(\mathcal{P}),$$

written infix as $p\circ q$.

---

[28]For instance, $\mathcal{D} \in \mathsf{inDi}\forall$ has a σ-action, and this is reflected dually as an $\mho$-action on prime filters.
Similarly, $\mathcal{D}$ has a permutation action; permutations are invertible, so the dual structure on prime filters is …another permutation action.

An $\reflectbox{$\sigma$}\circ$-algebra is just an $\reflectbox{$\sigma$}$-algebra with an equivariant *combination* function $\circ$ mapping pairs of points to sets of points. We now outline how the notion of spectral space is enriched by assuming $\circ$ (Definition 9.2.1) on its points.

We extend Definition 7.1.1:

DEFINITION 9.2.2. A **nominal $\sigma\circ$-topological space** $\mathcal{T}$ is nominal σ-topological space (Definition 7.1.1) whose points have the additional structure of a combination operator $\circ$ (Definition 9.2.1).

9.2.2. $\bullet$ and $-\!\!\bullet$, and spectral spaces. Extend Definition 3.4.1 as follows:

DEFINITION 9.2.3. Suppose $\mathcal{P} = (|\mathcal{P}|, \cdot, \mathcal{P}^{\partial}, \mathsf{amgis}, \circ)$ is an $\reflectbox{$\sigma$}\circ$-algebra. Give subsets $X, Y \subseteq |\mathcal{P}|$ **pointwise** actions as follows:

$$X\bullet Y = \bigcup\{p\circ q \mid p \in X,\ q \in Y\} \qquad Y-\!\!\bullet X = \{p \mid \forall q{\in}Y.p\circ q \subseteq X\}$$

DEFINITION 9.2.4. Continuing the notation of Definition 9.2.3, define $p\circ Y$ by

$$p\circ Y = \bigcup\{p\circ q \mid q \in Y\}.$$

LEMMA 9.2.5. *$Y-\!\!\bullet X$ can be conveniently rewritten using Definition 9.2.4 as*

$$Y-\!\!\bullet X = \{p \mid p\circ Y \subseteq X\}.$$

We can extend Proposition 3.4.2 to reflect the structure created by the combination action $\circ$:

PROPOSITION 9.2.6.

$$r \in X\bullet Y \Leftrightarrow \exists p{\in}X, q{\in}Y.r \in p\circ q \qquad r \in Y-\!\!\bullet X \Leftrightarrow r\circ Y \subseteq X \Leftrightarrow \forall q{\in}Y.r\circ q \subseteq X$$

We extend Definition 7.4.1 to account for the extra structure:

DEFINITION 9.2.7. Call a nominal σ-topological space $\mathcal{T}$ with $\circ$ **coherent** when it is coherent in the sense of Definition 7.4.1 and in addition:

(5) If X and Y are open and compact (so $X, Y{\in}cpct(\mathcal{T})$) then so are $X\bullet Y$ and $Y-\!\!\bullet X$.
(6) If $X, Y \in cpct(\mathcal{T})$ then $(X\bullet Y)[a{\mapsto}u] = X[a{\mapsto}u]\bullet Y[a{\mapsto}u]$ (so $cpct(\mathcal{T})$ satisfies $(\sigma\bullet)$).
(7) If $X \in cpct(\mathcal{T})$ then $(\partial_{\mathcal{T}}b-\!\!\bullet X)[a{\mapsto}u] = \partial_{\mathcal{T}}b-\!\!\bullet(X[a{\mapsto}u])$ (so $cpct(\mathcal{T})$ satisfies $(\sigma-\!\!\bullet)$).

LEMMA 9.2.8. *Suppose $\mathcal{T}$ is a nominal $\sigma\circ$-topological space (Definition 9.2.2) and $X, Y \subseteq |\mathcal{T}|$. Then:*

— $\forall W{\in}opens(\mathcal{T}).\big(W \subseteq Y-\!\!\bullet X \Leftrightarrow W\bullet Y \subseteq X\big)$.
— *If furthermore $\mathcal{T}$ is coherent and $X, Y \in cpct(\mathcal{T})$, then $Y-\!\!\bullet X = \bigcup\{W \in cpct(\mathcal{T}) \mid W\bullet Y \subseteq X\}$.*

Proof. Consider $W \in opens(\mathcal{T})$ and $X, Y \subseteq |\mathcal{T}|$.

By Lemma 9.2.5 $q\circ Y \subseteq X$ for any $q \in Y$ ($q\circ Y$ is from Definition 9.2.4). It follows from Definition 9.2.3 that $W\bullet Y \subseteq X$. Conversely if $W\bullet Y \subseteq X$ then by Definition 9.2.3, $p\circ Y \subseteq X$ for every $p \in W$, and thus by Lemma 9.2.5 also $p \in Y-\!\!\bullet X$ for every $p \in W$. So $W \subseteq Y-\!\!\bullet X$.

By the first part, $\bigcup\{W \in opens(\mathcal{T}) \mid W\bullet Y \subseteq X\} \subseteq Y-\!\!\bullet X$. If in addition $\mathcal{T}$ is coherent then if $X, Y \in cpct(\mathcal{T})$ then $Y-\!\!\bullet X \in cpct(\mathcal{T})$ and so $Y-\!\!\bullet X$ is one of the W such that $W\bullet Y \subseteq X$, so $Y-\!\!\bullet X \subseteq \bigcup\{W \in opens(\mathcal{T}) \mid W\bullet Y \subseteq X\}$. $\square$

Definition 9.2.9 extends Definition 7.8.1:

DEFINITION 9.2.9. A **nominal spectral space with** $\circ$ is a nominal $\sigma\circ$-topological space $\mathcal{T}$ (Definition 9.2.2) that is impredicative (7.5.2), coherent (Definition 9.2.7), and sober (Definition 7.7.2).

DEFINITION 9.2.10. We extend the notion of **morphism** $g : \mathcal{T}' \to \mathcal{T}$ of nominal spectral spaces from Definition 8.1.1 to insist that the inverse image g^{-1} should commute with $\bullet$ and $-\!\!\bullet$ in the following sense:

— If $X, Y \in cpct(\mathcal{T})$ then $g^{-1}(X \bullet Y) = g^{-1}(X) \bullet g^{-1}(Y)$.
— If $X \in cpct(\mathcal{T})$ then $g^{-1}(\partial b \multimap\!\!\bullet X) = \partial b \multimap\!\!\bullet g^{-1}(X)$.

Write $\mathsf{inSpect}\forall_\bullet$ for the category of nominal spectral spaces with $\circ$ (Definition 9.2.9), and morphisms between them whose inverse image functions commute with $\bullet$ and $\multimap\!\!\bullet$, as described above.

9.2.3. Useful technical lemmas. We conclude with a pair of useful technical lemmas:

Lemma 9.2.11.— $(\bigcup_i X_i) \bullet (\bigcup_j Y_j) = \bigcup_{ij}(X_i \bullet Y_j)$.
— $\varnothing \bullet Y = \varnothing = X \bullet \varnothing$.
— *If $X \subseteq X'$ and $Y \subseteq Y'$ then $X \bullet Y \subseteq X' \bullet Y'$.*

Proof. By elementary sets calculations on Definition 9.2.3. □

Lemma 9.2.12. *The adjoint axioms ($\bullet\epsilon$) and ($\bullet\eta$) from Definition 9.1.1 are valid. That is, for $X, Y \subseteq |\mathcal{P}|$:*

$$(U \multimap\!\!\bullet X) \bullet U \subseteq X \quad \textit{and} \quad X \subseteq U \multimap\!\!\bullet (X \bullet U)$$

Proof. Suppose $r \in (U \multimap\!\!\bullet X) \bullet U$. By Proposition 9.2.6 there exist $p \in U \multimap\!\!\bullet X$ and $q \in U$ such that $r \in p \circ q$, and by Proposition 9.2.6 again, $p \circ U \subseteq X$. Therefore by Definition 9.2.4 $p \circ q \subseteq X$ and in particular $r \in X$ as required.

Now consider $p \in X$. By Proposition 9.2.6 it suffices to show that $p \circ U \subseteq X \bullet U$. But this is immediate from Definitions 9.2.3 and 9.2.4. □

9.3. Filters in the presence of $\bullet$ and $\multimap\!\!\bullet$

The notions of filter and ideal from Definitions 6.1.1 and 6.1.4 do not change with the addition of $\bullet$ and $\multimap\!\!\bullet$. This is very convenient, because it leaves unaffected the 'logical' structure studied previously to Section 9 and the theorems we proved so far, still hold.

However, the addition of $\bullet$ and $\multimap\!\!\bullet$ adds structure, which gives us useful new ways to build extra filters, which we explore in Lemmas 9.3.3 and 9.3.4 and in Proposition 9.3.7.

Fix some $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ (Definition 9.1.6).

Definition 9.3.1. If q is a filter in $\mathcal{D}$ then define $q \bullet x$ by

$$q \bullet x = \{y \bullet x \mid y \in q\}.$$

$q \bullet x$ is not necessarily a filter, but the notation will be useful nonetheless.

Remark 9.3.2. It may be worth alerting the reader now to some notation we will shortly define. We considered $q \bullet x$; there will also be $q \circ x$ in Definition 9.3.5 and $p \bullet q$ in Definition 9.4.3. Of these, $p \bullet q$ is a fair generalisation of $p \bullet x$. In contrast, $q \circ x$ is something rather different.

Lemmas 9.3.3 and 9.3.4 clearly belong to the same family as Proposition 6.1.12. We need them for the proof of Theorem 9.4.7, where at a certain point we have assumed $r \in (x \bullet y)^\bullet$; we will use Lemmas 9.3.3 and 9.3.4 to generate prime filters p and q with $x \in p$ and $y \in q$, and we use that to conclude that $r \in x^\bullet \bullet y^\bullet$.

Lemma 9.3.3. *Suppose q and r are filters in $\mathcal{D}$ and x is an element in $\mathcal{D}$. Suppose r is prime and suppose q is a maximal filter such that $q \bullet x \subseteq r$.*

Then q is prime.

Proof. Suppose $z \vee z' \in q$ and $z, z' \notin q$.

By maximality $(q{+}z) \bullet x \not\subseteq r$ and $(q{+}z') \bullet x \not\subseteq r$. It follows that there exist $y \in q$ such that $(y \wedge z) \bullet x \notin r$ and $y' \in q'$ such that $(y' \wedge z') \bullet x \notin r$. Since r is prime, it follows that

$$((y \wedge z) \bullet x) \vee ((y' \wedge z') \bullet x) \notin r, \quad \text{and so} \quad u = ((y \wedge z) \vee (y' \wedge z')) \bullet x \notin r.$$

Now we rearrange this to

$$u = ((y \vee y') \wedge (z \vee y') \wedge (z' \vee y) \wedge (z \vee z')) \bullet x \notin r.$$

Now by assumption $y \vee y' \in q$ (since $y \in q$, and indeed also $y' \in q$), and $z \vee y' \in q$ (since $y' \in q$) and $y \vee z' \in q$ (since $y \in q$) and $z \vee z' \in q$. Since we assumed that $q \bullet x \subseteq r$, it follows that $u \in r$, a contradiction. □

LEMMA 9.3.4. *Suppose q, p, and r are filters in $\mathcal{D}$. Suppose r is prime and suppose q is maximal such that $p \bullet q \subseteq r$.*

Then q is prime.

Proof. Suppose $z \vee z' \in q$ and $z, z' \notin q$.

By maximality $p \bullet (q + z) \not\subseteq r$ and $p \bullet (q + z') \not\subseteq r$. It follows that there exist $x \in p$ and $y \in q$ such that $x \bullet (y \wedge z) \notin r$, and $x' \in p$ and $y' \in q$ such that $x' \bullet (y' \wedge z') \notin r$.

Since p is a filter, $x \wedge x' \in p$. For simplicity write $x'' = x \wedge x'$. By part 1 of Lemma 9.1.7 and condition 2 of Definition 6.1.1 for r it follows that

$$x'' \bullet (y \wedge z) \notin r \quad \text{and} \quad x'' \bullet (y' \wedge z') \notin r$$

and since r is prime we have

$$(x'' \bullet (y \wedge z)) \vee (x'' \bullet (y' \wedge z')) \notin r,$$

and so using $(\bullet \vee)$ and writing the left-hand-side as u we have

$$u = x'' \bullet ((y \wedge z) \vee (y' \wedge z')) \notin r.$$

Now we rearrange this to

$$u = x'' \bullet ((y \vee y') \wedge (y \vee z') \wedge (y' \vee z) \wedge (z \vee z')) \notin r.$$

By assumption $y \vee y'$, $y \vee z'$, $y' \vee z$, $z \vee z' \in q$ and by assumption $x'' \in p$. Furthermore by assumption $p \bullet q \subseteq r$. It follows that $u \in r$, a contradiction. □

DEFINITION 9.3.5. Suppose q is a filter in $\mathcal{D}$ and $y \in |\mathcal{D}|$. Define $q \circ y$ by

$$\boxed{q \circ y = \{x \mid y \multimap\!\!\bullet\, x \in q\}}$$

A justification for the notation $q \circ y$ is Lemma 9.3.6, which exhibits $q \circ y$ as a kind of dual to $y \multimap\!\!\bullet\, x$:

LEMMA 9.3.6. *$x \in q \circ y$ if and only if $y \multimap\!\!\bullet\, x \in q$.*

Proof. Routine from Definition 9.3.5. □

PROPOSITION 9.3.7. *If q is a filter in $\mathcal{D}$ then $q \circ y$ satisfies conditions 2, 3, and 4 of Definition 6.1.1.*

Proof. We consider the conditions in Definition 6.1.1 for $q \circ x$, freely using Lemma 9.3.6:

(2) *If $x \in q \circ y$ and $x \leq x'$ then $x' \in q \circ y$.* If $x \in q \circ y$ then $y \multimap\!\!\bullet\, x \in q$. By part 2 of Lemma 9.1.7 and condition 2 of Definition 6.1.1 for q we have $y \multimap\!\!\bullet\, x' \in q$, and so $x' \in q \circ y$.

(3) *If $x \in q \circ y$ and $x' \in q \circ y$ then $x \wedge x' \in q \circ y$.* Suppose $x, x' \in q \circ y$, so that $y \multimap\!\!\bullet\, x, y \multimap\!\!\bullet\, x' \in q$. Therefore $(y \multimap\!\!\bullet\, x) \wedge (y \multimap\!\!\bullet\, x') \in q$ and by $(\multimap\!\!\bullet \wedge)$ $y \multimap\!\!\bullet\, (x \wedge x') \in q$. Thus $x \wedge x' \in q$.

(4) *If $\text{И}b.(b\ a) \cdot x \in q \circ y$ then $\forall a.x \in q \circ y$.* We note that for fresh a' (so $a' \# x, y$)[29] if $b \# x$ then by Corollary 2.1.10 $(b\ a) \cdot x = (b\ a') \cdot ((a'\ a) \cdot x)$ and by Lemma 4.1.3 $\forall a.x = \forall a'.(a'\ a) \cdot x$. So using Proposition 2.3.3 we may assume without loss of generality that $a \# y$.

Now suppose $\text{И}b.(b\ a) \cdot x \in q \circ y$. By Lemma 9.3.6 $\text{И}b.(y \multimap\!\!\bullet\, (b\ a) \cdot x \in q)$, and by Lemma 2.3.5 (for the equivariant function $\multimap\!\!\bullet$) $\text{И}b.(b\ a) \cdot (y \multimap\!\!\bullet\, x) \in q$. By condition 4 of Definition 6.1.1 $\forall a.(y \multimap\!\!\bullet\, x) \in$

[29]We do not assume that q has finite support so we do not know that $a' \# q$, but that will not be a problem.

q, thus by $(\multimap\!\!\bullet\forall)$ and condition 2 of Definition 6.1.1 $y\multimap\!\!\bullet\forall a.x \in q$ and so by Lemma 9.3.6 $\forall a.x \in q\circ y$.

□

So $q\circ y$ might fail to be a filter if $y\multimap\!\!\bullet\bot \in q$, but if $\bot \notin q\circ y$ then $q\circ y$ is indeed a filter.

9.4. The second representation theorem

We now show how to build N-algebras with $\circ$ (recall from Subsection 9.2 that $\circ$ is the topological dual to application $\bullet$).

For the rest of this subsection fix some $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ (Definition 9.1.6). Recall the definition of $x^\bullet$ from Definition 6.3.1.

The first two clauses of Definition 9.4.1 echo Definition 3.3.1; the third gives prime filters a $\circ$ action in the sense of Definition 9.2.1:

Definition 9.4.1. Give prime filters p and q in $\mathcal{D}$ actions as follows:

$$\pi{\cdot}p = \{\pi{\cdot}x \mid x \in p\} \qquad\qquad p[u{\leftarrow}a] = \{x \mid x[a{\mapsto}u] \in p\}$$
$$p\circ q = \bigcap\{(x\bullet y)^\bullet \mid x \in p, y \in q\}$$

Proposition 9.4.2. *Prime filters of $\mathcal{D}$ form an N-algebra with $\circ$ in the sense of Definition 9.2.1.*

Proof. Equivariance of $\circ$ is immediate from Theorem 2.3.1. We just use Proposition 6.2.3. □

We now work towards Proposition 9.4.6, which will be needed for Theorem 9.4.7. Definition 9.4.3 will be useful:

Definition 9.4.3. Suppose p and q are prime filters in $\mathcal{D}$ and $y \in |\mathcal{D}|$. Define $p\bullet q$ by

$$p\bullet q = \{x\bullet y \mid x \in p,\ y \in q\}$$

$p\bullet q$ is a useful technical notation for Proposition 9.4.4, which extends Proposition 3.3.2:

Proposition 9.4.4. *Suppose p is a prime filter in $\mathcal{D}$. Then:*

(1) $r \in p\circ q$ *if and only if* $p\bullet q \subseteq r$.
(2) $r \in p\circ Y$ *if and only if* $\exists q{\in}Y.p\bullet q \subseteq r$.

Proof. By easy calculations on Definitions 9.4.1 and 9.2.4. □

It will be convenient to package a few definitions and part of Proposition 9.4.4 into a technical corollary:

Corollary 9.4.5. $r \in p\circ y^\bullet$ *if and only if* $\exists q.(y{\in}q \wedge p\bullet q \subseteq r)$.

Proof. We unpack the definition of $p\circ y^\bullet$ in Definition 9.2.4, and note from Definition 6.3.1 that $q \in y^\bullet$ if and only if $y \in q$. We use part 2 of Proposition 9.4.4. □

Proposition 9.4.6. *Suppose r is a prime filter in $\mathcal{D}$. Then $r\circ y^\bullet \subseteq x^\bullet$ if and only if $y\multimap\!\!\bullet x \in r$.*

Proof. We prove two implications.

The right-to-left implication. Suppose $y\multimap\!\!\bullet x \in r$ and suppose $p \in r\circ y^\bullet$. We need to show that $p \in x^\bullet$.

By Corollary 9.4.5 $p \in r\circ y^\bullet$ means that for some q with $y \in q$ it is the case that $r\bullet q \subseteq p$. So given some such q, since $y\multimap\!\!\bullet x \in r$ we have that $(y\multimap\!\!\bullet x)\bullet y \in p$. By $(\bullet\epsilon)$ $(y\multimap\!\!\bullet x)\bullet y \leq x$ and since p is up-closed (condition 2 of Definition 6.1.1) we have $x \in p$, thus by Definition 6.3.1 $p \in x^\bullet$ as required.

The left-to-right implication. Suppose $r\circ y^\bullet \subseteq x^\bullet$; so if p is a prime filter and $p \in r\circ y^\bullet$, then $x \in p$. We need to show that $y\multimap\!\!\bullet x \in r$.

Consider $r\circ y$ from Definition 9.3.5. If $r\circ y$ is not a filter then by Proposition 9.3.7 $\bot \in r\circ y$ and so by Lemma 9.3.6 $y\multimap\!\!\bullet\bot \in r$. Thus by Lemma 9.1.7 $y\multimap\!\!\bullet x \in r$ as required.

So now suppose $r\circ y$ is a filter. By Corollary 9.4.5 $p \in r\circ y^\bullet$ when for some q with $y \in q$ we have $r \bullet q \subseteq p$.

Take p to be any prime filter containing $r\circ y$; at least one such p exists, by Theorem 6.1.13. By Corollary 6.1.8 $y\!\uparrow$ is a filter, and from Lemma 9.3.4 there exists a prime filter q with $y\!\uparrow \subseteq q$ and $r \bullet q \subseteq p$. Also $y \in q$. So we have some q with $y \in q$ and $r \bullet q \subseteq p$, so by assumption $x \in p$.

Now the reasoning of the previous paragraph holds for *any* prime p containing $r\circ y$. That is, for any prime p with $r\circ y \subseteq p$ it is the case that $x \in p$.

It follows by Lemma 6.3.5 that $x \in r\circ y$, and so by Lemma 9.3.6 that $y \multimap\!\bullet\, x \in r$ as required. $\square$

Theorem 9.4.7 extends Lemma 6.4.2 for the additional structure of $\bullet$ and $\multimap\!\bullet$:

THEOREM 9.4.7. *Suppose $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ and $x, y \in |\mathcal{D}|$. Then:*

(1) $x^\bullet \bullet y^\bullet = (x \bullet y)^\bullet$.
(2) $y^\bullet \multimap\!\bullet\, x^\bullet = (y \multimap\!\bullet\, x)^\bullet$.

Proof. We consider each part in turn.

By Proposition 9.2.6 $r \in x^\bullet \bullet y^\bullet$ if and only if $\exists p, q.(x{\in}p \wedge y{\in}q) \wedge r \in p\circ q$, and by Proposition 9.4.4 this is if and only if $\exists p, q.(x{\in}p \wedge y{\in}q) \wedge p \bullet q {\subseteq} r$.

So suppose there exist prime filters p and q with $x{\in}p$ and $y{\in}q$ and $p\bullet q \subseteq r$. Then clearly $x \bullet y \in r$.

Conversely suppose $x \bullet y \in r$ for some prime filter r. By Corollary 6.1.8 $x\!\uparrow$ is a filter and it follows from Lemma 9.3.3 that there exists a prime filter p with $x \in p$ and $p \bullet y \subseteq r$. Also by Corollary 6.1.8 $y\!\uparrow$ is a filter and from Lemma 9.3.4 there exists a prime filter q with $y \in q$ and $p \bullet q \subseteq r$.

For the second part we reason as follows, using Proposition 9.2.6 and Proposition 9.4.6:

$$r \in y^\bullet \multimap\!\bullet\, x^\bullet \overset{\text{Prop 9.2.6}}{\Leftrightarrow} r\circ y^\bullet \subseteq x^\bullet \overset{\text{Prop 9.4.6}}{\Leftrightarrow} r \in (y \multimap\!\bullet\, x)^\bullet$$

$\square$

We can now easily extend Definition 6.4.3 (the definition of $\mathcal{D}^\bullet$):

DEFINITION 9.4.8. As in Definition 6.4.3 we take $\mathcal{D}^\bullet$ to have:

— $|\mathcal{D}^\bullet| = \{x^\bullet \mid x \in |\mathcal{D}|\}$
— $(\mathcal{D}^\bullet)^\partial = \mathcal{D}^\partial$ and $\partial_{\mathcal{D}^\bullet} u = (\partial_{\mathcal{D}} u)^\bullet$
— $\pi{\cdot}(x^\bullet) = (\pi{\cdot}x)^\bullet$
— $x^\bullet[a{\mapsto}u] = (x[a{\mapsto}u])^\bullet$

We give $\mathcal{D}^\bullet$ the actions $X \bullet Y$ and $Y \multimap\!\bullet\, X$ from Definition 9.2.3.

So Definitions 6.4.3 and 9.4.8 overload the notation $\mathcal{D}^\bullet$ for "the object in $\mathsf{inDi}\forall$ composed of prime filters of $\mathcal{D} \in \mathsf{inDi}\forall$" and "the object of $\mathsf{inDi}\forall_\bullet$ composed of prime filters of $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$". The meaning will always be clear.

LEMMA 9.4.9. *Suppose $x, y \in |\mathcal{D}|$ and $u \in |\mathcal{D}^\partial|$. Then $(\sigma\bullet)$ and $(\sigma\multimap\!\bullet)$ are valid in $\mathcal{D}^\bullet$:*

$$(x^\bullet \bullet y^\bullet)[a{\mapsto}u] = x^\bullet[a{\mapsto}u] \bullet (y^\bullet[a{\mapsto}u])$$
$$(\partial b \multimap\!\bullet\, x^\bullet)[a{\mapsto}u] = \partial b \multimap\!\bullet\, (x^\bullet[a{\mapsto}u])$$

Proof. We reason as follows:

$$\begin{aligned}
(x^\bullet \bullet y^\bullet)[a\mapsto u] &= (x\bullet y)^\bullet[a\mapsto u] && \text{Theorem 9.4.7}\\
&= ((x\bullet y)[a\mapsto u])^\bullet && \text{Lemma 6.4.1}\\
&= (x[a\mapsto u]\bullet y[a\mapsto u])^\bullet && (\sigma\bullet)\\
&= (x[a\mapsto u])^\bullet \bullet (y[a\mapsto u])^\bullet && \text{Theorem 9.4.7}\\
&= x^\bullet[a\mapsto u]\bullet y^\bullet[a\mapsto u] && \text{Lemma 6.4.1}
\end{aligned}$$

$$\begin{aligned}
(\partial_{\mathcal{D}^\bullet} b \multimap\!\bullet\, x^\bullet)[a\mapsto u] &= (\partial_{\mathcal{D}} b \multimap\!\bullet\, x)^\bullet[a\mapsto u] && \text{Theorem 9.4.7}\\
&= (\partial_{\mathcal{D}} b \multimap\!\bullet\, x)[a\mapsto u]^\bullet && \text{Lemma 6.4.1}\\
&= (\partial_{\mathcal{D}} b \multimap\!\bullet\, x[a\mapsto u])^\bullet && (\sigma\multimap\!\bullet)\\
&= \partial_{\mathcal{D}^\bullet} b \multimap\!\bullet\, (x[a\mapsto u])^\bullet && \text{Theorem 9.4.7}
\end{aligned}$$

□

LEMMA 9.4.10. *The adjoint and compatibility axioms from Definition 9.1.1 are valid in $\mathcal{D}^\bullet$. That is:*

$$\begin{array}{lll}
(\bullet\epsilon) & (u^\bullet \multimap\!\bullet\, x^\bullet)\bullet x^\bullet \subseteq x^\bullet & \\
(\bullet\eta) & x^\bullet \subseteq u^\bullet \multimap\!\bullet\, (x^\bullet\bullet u^\bullet) & \\
(\bullet\bot) & \varnothing\bullet u^\bullet = \varnothing & x^\bullet\bullet\varnothing = \varnothing\\
(\bullet\wedge) & (x^\bullet\cap y^\bullet)\bullet u^\bullet \subseteq (x^\bullet\bullet u^\bullet)\cap(y^\bullet\bullet u^\bullet) & x^\bullet\bullet(u^\bullet\cap v^\bullet) \subseteq (x^\bullet\bullet u^\bullet)\cap(x^\bullet\bullet v^\bullet)\\
(\bullet\vee) & (x^\bullet\cup y^\bullet)\bullet u = (x^\bullet\bullet u^\bullet)\cup(y^\bullet\bullet u^\bullet) & x^\bullet\bullet(u^\bullet\cup v^\bullet) = (x^\bullet\bullet u^\bullet)\cup(x^\bullet\bullet v^\bullet)\\
 & \dots & \\
(\multimap\!\bullet\forall) & b\#u^\bullet \Rightarrow \bigcap^{\#b}(u^\bullet \multimap\!\bullet\, x^\bullet) \subseteq u^\bullet \multimap\!\bullet\, (\bigcap^{\#b} x^\bullet) &
\end{array}$$

Proof. The easiest proof is to combine Theorem 9.4.7 and Lemma 6.4.2 with Lemma 6.3.2 and with the relevant axiom for $\mathcal{D}$.

In the cases of $(\bullet\forall)$ and $(\multimap\!\bullet\forall)$ which have a freshness condition, we use Lemma 2.3.4 to choose a suitably fresh representative of $u^\bullet$ (i.e. a u' such that $(u')^\bullet = u^\bullet$ and $b\#u'$).[30] □

Recall the definition of $\mathsf{inDi}\forall_\bullet$ from Definition 9.1.6.

THEOREM 9.4.11 (Second representation theorem). *If $\mathcal{D}\in\mathsf{inDi}\forall_\bullet$ then $\mathcal{D}^\bullet$ from Definition 9.4.8 is in $\mathsf{inDi}\forall_\bullet$, and the assignment $x\mapsto x^\bullet$ is an isomorphism in $\mathsf{inDi}\forall_\bullet$ between $\mathcal{D}$ and $\mathcal{D}^\bullet$.*

Proof. Theorem 6.4.4 handles the purely logical structure ($\cap$, $\varnothing$, $\cup$, and $\bigcap^{\#a}$). Lemmas 9.4.9 and 9.4.10 validate the axioms for $\bullet$ and $\multimap\!\bullet$. □

9.5. Construction of the topological space $F(\mathcal{D})$, with $\bullet$ and $\multimap\!\bullet$

Recall the definitions of $\mathsf{inDi}\forall_\bullet$ and $\mathsf{inSpect}\forall_\bullet$ from Definitions 9.1.6 and 9.2.10. We extend Definition 7.2.1:

DEFINITION 9.5.1. Suppose $\mathcal{D}\in\mathsf{inDi}\forall_\bullet$. Define $F(\mathcal{D})\in\mathsf{inSpect}\forall_\bullet$ by:

(1) $|F(\mathcal{D})| = |points(\mathcal{D})|$ and $F(\mathcal{D})^\partial = \mathcal{D}^\partial$.
(2) $\pi\cdot p = \{\pi\cdot x \mid x\in p\}$.
(3) $p[u\leftarrow\!\!\shortmid a] = \{x \mid x[a\mapsto u]\in p\}$.
(4) $p\circ q = \bigcap\{(x\bullet y)^\bullet \mid x\in p, y\in q\}$, following Definition 9.4.1.
(5) $opens(F(\mathcal{D}))$ is the closure of $\{x^\bullet \mid x\in|\mathcal{D}|\}$ under strictly finitely supported unions.
(6) $\partial_{F(\mathcal{D})}$ maps $u\in|\mathcal{D}^\partial|$ to $(\partial_{\mathcal{D}} u)^\bullet$.[31]

[30] Lemma 2.3.4 is applied here to the function $u\mapsto u^\bullet$. Equivariance is defined in Definition 2.1.6; Definition 9.4.8 states (amongst other things) that this function is equivariant.

[31] It might be helpful to unwind the definitions for this final clause. This is not complicated—it just takes in a lot of definitions! $u\in|\mathcal{D}^\partial|$ is an element of the termlike σ-algebra over which $F(\mathcal{D})$ has an $\amalg$-action; $\partial_{\mathcal{D}} u\in|\mathcal{D}|$ is an element of $\mathcal{D}$ the (impredicative) nominal distributive lattice with $\forall$ and $\bullet$; $(\partial_{\mathcal{D}} u)^\bullet$ is the set of prime filters in $points(\mathcal{D})$ that contain $\partial_{\mathcal{D}} u$. By Proposition 7.3.8 this set of prime filters is compact, that is it is in $cpct(\mathcal{T})$, as required in Definition 7.5.2.

In Definition 9.5.1 we claim that $F(\mathcal{D})$ is a nominal spectral space with $\circ$ (Definition 9.2.9). This needs to be proved: Theorem 9.5.3 assembles the various verifications.

Recall from Definition 8.1.5 the map from $f : \mathcal{D} \to \mathcal{D}'$ to $F(f) : F(\mathcal{D}') \to F(\mathcal{D})$.

PROPOSITION 9.5.2. *If $f : \mathcal{D} \to \mathcal{D}'$ is in* $\mathsf{inDi}\forall_\bullet$ *then $F(f)$ commutes with $\bullet$ and $\multimap$ as specified in Definition 9.5.1. That is, using Propositions 7.3.8 and 7.3.10:*

(1) $F(f)^{\text{-}1}(x^\bullet \bullet y^\bullet) = F(f)^{\text{-}1}(x^\bullet) \bullet F(f)^{\text{-}1}(y^\bullet)$
(2) $F(f)^{\text{-}1}(\partial b \multimap x^\bullet) = \partial b \multimap F(f)^{\text{-}1}(x^\bullet)$.

As a corollary, if $f : \mathcal{D} \to \mathcal{D}'$ then $F(f) : F(\mathcal{D}') \to F(\mathcal{D})$ is a morphism in the sense of Definition 9.2.10.

Proof. The corollary follows direct from Definition 9.2.10 using Theorem 8.1.9.

For the first part, we reason as follows:

$$\begin{array}{rll}
r \in F(f)^{\text{-}1}(x^\bullet \bullet y^\bullet) & \Leftrightarrow F(f)(r) \in x^\bullet \bullet y^\bullet & \text{Inverse image} \\
& \Leftrightarrow F(f)(r) \in (x \bullet y)^\bullet & \text{Theorem 9.4.7} \\
& \Leftrightarrow x \bullet y \in F(f)(r) & \text{Definition 6.3.1} \\
& \Leftrightarrow f(x \bullet y) \in r & \text{Definition 8.1.5} \\
& \Leftrightarrow f(x) \bullet f(y) \in r & \text{Definition 9.1.6} \\
& \Leftrightarrow r \in (f(x) \bullet f(y))^\bullet & \text{Definition 6.3.1} \\
& \Leftrightarrow r \in f(x)^\bullet \bullet f(y)^\bullet & \text{Theorem 9.4.7} \\
& \Leftrightarrow r \in F(f)^{\text{-}1}(x^\bullet) \bullet F(f)^{\text{-}1}(y) & \text{Lemma 8.1.6}
\end{array}$$

The reasoning for the second part, for $\multimap$, is similar:

$$\begin{array}{rll}
r \in F(f)^{\text{-}1}(\partial_{F(\mathcal{D})} b \multimap x^\bullet) & \Leftrightarrow F(f)(r) \in \partial_{F(\mathcal{D})} b \multimap x^\bullet & \text{Inverse image} \\
& \Leftrightarrow F(f)(r) \in (\partial_{\mathcal{D}} b \multimap x)^\bullet & \text{Theorem 9.4.7} \\
& \Leftrightarrow \partial_{\mathcal{D}} b \multimap x \in F(f)(r) & \text{Definition 6.3.1} \\
& \Leftrightarrow f(\partial_{\mathcal{D}} b \multimap x) \in r & \text{Definition 8.1.5} \\
& \Leftrightarrow \partial_{\mathcal{D}'} b \multimap f(x) \in r & \text{Definition 9.1.6} \\
& \Leftrightarrow r \in (\partial_{\mathcal{D}'} b \multimap f(x))^\bullet & \text{Definition 6.3.1} \\
& \Leftrightarrow r \in (\partial_{\mathcal{D}'} b)^\bullet \multimap f(x)^\bullet & \text{Theorem 9.4.7} \\
& \Leftrightarrow r \in \partial_{F(\mathcal{D}')} b \multimap F(f)^{\text{-}1}(x) & \text{Lemma 8.1.6}
\end{array}$$

$\square$

Theorem 9.5.3 extends Theorem 8.1.9 from $\mathsf{inDi}\forall$ and $\mathsf{inSpect}\forall$ to $\mathsf{inDi}\forall_\bullet$ and $\mathsf{inSpect}\forall_\bullet$:

THEOREM 9.5.3. *F is a functor from* $\mathsf{inDi}\forall_\bullet$ *to* $\mathsf{inSpect}\forall_\bullet$.

Proof. This is mostly Theorem 8.1.9 combined with Proposition 9.5.2. We do also need to check the extra conditions on coherence from Definition 9.2.7; these follow easily from Theorem 9.4.7, Lemma 9.4.9, and Propositions 7.3.8 and 7.3.10. $\square$

Part 1 of Proposition 9.5.2 is stated only for sets of the form $x^\bullet$ and $y^\bullet$. In fact, we note that it can be extended to all $X, Y \in \mathit{opens}(F(\mathcal{D}))$:

COROLLARY 9.5.4. $F(f)^{\text{-}1}(X \bullet Y) = F(f)^{\text{-}1}(X) \bullet F(f)^{\text{-}1}(Y)$ *for all $X, Y \in \mathit{opens}(F(\mathcal{D}))$ (and not only $X, Y \in \mathit{cpct}(F(\mathcal{D}))$).*

Proof. By construction in Definition 7.2.1 every open set in $F(\mathcal{D})$ is a strictly finitely supported union of compact opens, which by Propositions 7.3.8 and 7.3.10 have in $F(\mathcal{D})$ the form $x^\bullet$ and $y^\bullet$ for $x, y \in |\mathcal{D}|$. We use Lemma 9.2.11 and Proposition 9.5.2. $\square$

Corollary 9.5.4 would not work for $\multimap$, because a result corresponding to Lemma 9.2.11 does not hold for it.

9.6. The duality, in the presence of $\bullet$ and $\multimap\!\!\bullet$

It is routine to extend Definition 7.6.1—which sends a spectral space $\mathfrak{T}$ to the lattice of its compact open sets $G(\mathfrak{T})$—and Definition 8.2.1—which sends a spectral map g to its inverse image function $G(g) = g^{-1}$—to the case where we also assume $\bullet$ and $-\!\!\bullet$. We write it out, just to be clear:

Definition 9.6.1. If $\mathfrak{T} \in \mathsf{inSpect}\forall_\bullet$ (Definition 9.2.9) define $G(\mathfrak{T}) \in \mathsf{inDi}\forall_\bullet$ (Definition 9.1.6) by:

— $|G(\mathfrak{T})| = cpct(\mathfrak{T})$ and $G(\mathfrak{T})^\partial = \mathfrak{T}^\partial$.
— $\pi\cdot U = \{\pi\cdot p \mid p \in U\}$ and $U[a\mapsto u] = \{p \mid p[u\leftarrow a] \in U\}$, where $U \in |G(\mathfrak{T})|$ and $u \in |G(\mathfrak{T})^\partial|$.
— $\top$, $\wedge$, $\bot$, $\vee$, and $\forall$ are interpreted as the whole underlying set, set intersection, the empty set, set union, and $\bigcap^{\#a}$.
— $X\bullet Y$ and $Y-\!\!\bullet X$ are interpreted as specified in Definition 9.2.3.

Given $g : \mathfrak{T} \to \mathfrak{T}' \in \mathsf{inSpect}\forall_\bullet$ from Definition 9.2.10, define $G(g) : G(\mathfrak{T}') \to G(\mathfrak{T})$ by $G(g)(U) = g^{-1}(U)$.

Lemma 9.6.2. *Continuing the notation of Definition 9.6.1, suppose $U, V \in |G(\mathfrak{T})|$ and suppose $\mathcal{U} \subseteq |G(\mathfrak{T})|$. Then*

$$\bigcap_{U\in\mathcal{U}} (V-\!\!\bullet U) = V-\!\!\bullet \bigcap_{U\in\mathcal{U}} \mathcal{U}.$$

Proof. Using Proposition 9.2.6 $r \in V-\!\!\bullet \bigcap\mathcal{U}$ if and only if $r\circ V \subseteq U$ for every $U \in \mathcal{U}$, and by Proposition 9.2.6 again this is if and only if $r \in V-\!\!\bullet U$ for every $U \in \mathcal{U}$, which is if and only if $r \in \bigcap_{U\in\mathcal{U}} V-\!\!\bullet U$. □

Lemma 9.6.3. *If $\mathfrak{T} \in \mathsf{inSpect}\forall_\bullet$ then $G(\mathfrak{T})$ validates the axioms from Definition 9.1.1.*

Proof. We consider each axiom in turn. We take $X, Y, X', U \in cpct(\mathfrak{T})$ (open compacts in $\mathfrak{T}$):

— *Axioms* $(\sigma\bullet)$ *and* $(\sigma-\!\!\bullet)$. By assumption in Definition 9.2.7.
— *The adjoint axioms* $(\bullet\epsilon)$ *and* $(\bullet\eta)$. Direct from Lemma 9.2.12.
— *Axioms* $(\bullet\bot)$ *and* $(\bullet\vee)$ …are Lemma 9.2.11.
— *Axioms* $(\bullet\wedge)$. If $r \in (X \cap X')\bullet Y$ then $r \in p\circ q$ for some $p \in X \cap X'$ and $q \in Y$. It follows that $r \in p\circ q$ for $p \in X$ and $q \in Y$ and $r \in p\circ q$ for $p \in X'$ and $q \in Y$, and therefore $r \in (X\bullet Y) \cap (X'\bullet Y)$. The second $(\bullet\wedge)$ axiom follows similarly.
— *Axiom* $(\bullet\forall)$. Suppose $b\#U$ and $r \in (\bigcap^{\#a} X)\bullet U$. It follows by Lemma 9.2.11 that $r \in X[a\mapsto w]\bullet U$ for every $w \in |\mathfrak{T}^\partial|$. Now by part 1 of Lemma 3.4.6 and condition 6 of Definition 9.2.7, $X[a\mapsto w]\bullet U = (X\bullet U)[a\mapsto w]$.
So $r \in (X\bullet U)[a\mapsto w]$ for every $w \in |\mathfrak{T}^\partial|$, and by line 2 of Proposition 5.2.8 also $r \in \bigcap^{\#a}(X\bullet U)$.
— *Axiom* $(-\!\!\bullet\wedge)$. From Lemma 9.6.2.
— *Axiom* $(-\!\!\bullet\vee)$. $r \in (U-\!\!\bullet X) \cup (U-\!\!\bullet X')$ means $r\circ U \subseteq X$ or $r\circ U \subseteq X'$. In either case, $r\circ U \subseteq X \cup X'$ and this means $r \in U-\!\!\bullet(X \cup X')$.
— *Axiom* $(-\!\!\bullet\forall)$. Suppose $b\#U$. We reason as follows:

$$\begin{array}{r@{\ }l@{\qquad}l}
\bigcap^{\#b}(U-\!\!\bullet X) & = \bigcap_{u\in|\mathfrak{T}^\partial|}(U-\!\!\bullet X)[b\mapsto u] & \text{Definition 5.2.1}\\
& \subseteq \bigcap_{u\in|\mathfrak{T}^\partial|} U[b\mapsto u]-\!\!\bullet X[b\mapsto u] & \text{Lemma 9.1.9}\\
& = \bigcap_{u\in|\mathfrak{T}^\partial|} U-\!\!\bullet X[b\mapsto u] & (\sigma\#),\ b\#U\\
& = U-\!\!\bullet \bigcap_{u\in|\mathfrak{T}^\partial|} X[b\mapsto u] & \text{Lemma 9.6.2}\\
& = U-\!\!\bullet \bigcap^{\#b} X & \text{Definition 5.2.1}
\end{array}$$

□

Proposition 9.6.4. *G from Definition 9.6.1 is a functor from $\mathsf{inSpect}\forall_\bullet^{op}$ to $\mathsf{inDi}\forall_\bullet$.*

Proof. The action on objects is handled by Theorem 7.6.2 and Lemma 9.6.3. The action on morphisms is handled by Proposition 8.2.2 and by the two conditions on g^{-1} in Definition 9.2.10. □

We checked in Proposition 9.5.2 that this is true for $g = F(f)^{-1}$, so we can extend Proposition 7.6.3:

PROPOSITION 9.6.5. *If* $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ *then* $GF(\mathcal{D})$ *is equal to* $\mathcal{D}^\bullet$ *from Definition 9.4.8, and the map* $x \mapsto x^\bullet$ *is an isomorphism in* $\mathsf{inDi}\forall_\bullet$.

Proof. Just as the proof of Proposition 7.6.3; the extra structure of $\bullet$ and $\multimap\!\bullet$ has no effect. We use Theorem 9.4.11. □

It is routine to check that the $\bullet$ and $\multimap\!\bullet$ structure is orthogonal to the material of Subsections 8.2 and 8.3, and so we obtain Theorem 9.6.6:

THEOREM 9.6.6 (The duality theorem). $G : \mathsf{inSpect}\forall_\bullet^{\mathsf{op}} \to \mathsf{inDi}\forall_\bullet$ *defines an equivalence between* $\mathsf{inDi}\forall_\bullet$ *and* $\mathsf{inSpect}\forall_\bullet^{op}$.

Theorem 9.6.6 exhibits $\mathsf{inDi}\forall_\bullet$ and $\mathsf{inSpect}\forall_\bullet$ as dual to one another. This is a general result—the abstract nominal algebra structures in $\mathsf{inDi}\forall_\bullet$ correspond dually to concrete topological spaces in $\mathsf{inSpect}\forall_\bullet$.

It remains to show how $\mathsf{inDi}\forall_\bullet$ and $\mathsf{inSpect}\forall_\bullet$ relate specifically to the untyped λ-calculus.

IV. APPLICATION TO THE λ-CALCULUS

10. THE λ-CALCULUS

In this section we sketch the untyped λ-calculus and show how it has been living inside $\mathsf{inDi}\forall_\bullet$ all along: this is $\boldsymbol{\lambda} a.x$ in Notation 10.2.1. We make formal that Notation 10.2.1 is 'a right thing to do' with Proposition 10.2.4, Definition 10.4.1, and Theorem 10.4.7.

We also briefly unpack what $\boldsymbol{\lambda} a.X$ is when X is an open set in the topological representations in $\mathsf{inSpect}\forall_\bullet$. This is Proposition 10.2.6.

Thus, we leverage our topological duality to give both abstract and concrete (i.e. nominal poset flavoured and nominal sets flavoured) semantics for the λ of the untyped λ-calculus.

10.1. Syntax of the λ-calculus

DEFINITION 10.1.1. Define **λ-terms** as usual by

$$s ::= a \mid \lambda a.s \mid s's$$

where a ranges over atoms (so we use atoms as variable symbols when we build our syntax, in nominal style).[32]

— We treat λ-terms as equal up to α-equivalence.[33]
— We assume capture-avoiding substitution $s[a{:=}u]$.
— We write $fa(s)$ for the free atoms (variables) of s.

DEFINITION 10.1.2. Consider λ-terms as a nominal set (Definition 2.1.5) by giving them the natural permutation action:

$$\pi{\cdot}a = \pi(a) \qquad \pi{\cdot}(\lambda a.s) = \lambda\pi(a).\pi{\cdot}s \qquad \pi{\cdot}(s's) = (\pi{\cdot}s')(\pi{\cdot}s)$$

Write LmTm for the nominal set of λ-terms with this permutation action.

It is a fact that with the permutation action above, $fa(s)$ the free atoms of s and $supp(s)$ the atoms in the support of s, coincide.

DEFINITION 10.1.3. Consider λ-terms as a termlike σ-algebra (Definition 3.1.1) by setting

$$s[a{\mapsto}u] = s[a{:=}u],$$

[32] We could allow constants c too, if we wished.

[33] ...using nominal abstract syntax [Gab01; GP01] or by taking equivalence classes or by whatever other method the reader prefers.

so that $[a{\mapsto}u]$ acting on s is 's with a substituted for u'.

It is a fact that this does indeed determine a termlike σ-algebra. The nominal algebra axioms of Figure 1 reflect valid properties of capture-avoiding substitution on λ-terms.

10.2. λ, β, and η using adjoints

In objects of $\mathsf{inDi}\forall_\bullet$, λ-abstraction arises naturally by combining the 'logical' structure $\forall$ and $\leq$ with the 'combinational' structure of $\bullet$ and $-\!\!\bullet$; this is Notation 10.2.1. We shall see that β-reduction and η-expansion arise as natural corollaries of the adjoint properties of $\bullet$ and $-\!\!\bullet$; this is Proposition 10.2.4.

Proposition 10.2.6 unpacks what this means in $F(\mathcal{D})$, and Definition 10.4.1 and Theorem 10.4.7 show how we can interpert the full untyped λ-calculus.

10.2.1. λ using $\forall$ and $-\!\!\bullet$.

NOTATION 10.2.1. Suppose $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ and $x \in |\mathcal{D}|$. Write $\boldsymbol{\lambda} a.x$ for $\forall a.(\partial a -\!\!\bullet x)$.

REMARK 10.2.2. We unpack some of Notation 10.2.1. The notation ∂a is explained in detail in Notation 4.5.3. In full,

$$\boldsymbol{\lambda} a.x \quad\text{and}\quad \forall a.(\partial a -\!\!\bullet x) \qquad\text{mean}\qquad \forall_{\mathcal{D}} a.(\partial_{\mathcal{D}}(a_{\mathcal{D}^\partial}) -\!\!\bullet_{\mathcal{D}} x).$$

Here $a_{\mathcal{D}^\partial}$ is the copy of a in the termlike σ-algebra $\mathcal{D}^\partial$ and $\partial_{\mathcal{D}}$ maps this to $|\mathcal{D}|$.

$\forall a$ is from Definition 4.1.2. $-\!\!\bullet$ is a right adjoint to application and is from Definition 9.1.1.

LEMMA 10.2.3. *If $b\#u$ then $(\boldsymbol{\lambda} b.x)[a{\mapsto}u] = \boldsymbol{\lambda} b.(x[a{\mapsto}u])$.*

Proof. We unpack Notation 10.2.1. By assumption in Definition 4.4.3 the σ-action is compatible (Definition 4.3.1) so $(\forall b.\partial b -\!\!\bullet x)[a{\mapsto}u] = \forall b.((\partial b -\!\!\bullet x)[a{\mapsto}u])$. We use $(\sigma -\!\!\bullet)$ from Definition 9.1.1 and $(\sigma\#)$ (since $b\#a$). □

We now derive β-reduction and η-expansion from the counit and unit axioms $(\bullet\epsilon)$ and $(\bullet\eta)$ respectively:

PROPOSITION 10.2.4. *Suppose $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$, $x \in |\mathcal{D}|$, $u \in |\mathcal{D}^\partial|$, and a is an atom. Then:*

(1) $(\boldsymbol{\lambda} a.x)\bullet\partial u \leq x[a{\mapsto}u]$.
(2) If $a\#x$ then $x \leq \boldsymbol{\lambda} a.(x\bullet\partial a)$ (∂u and ∂a from Notation 4.5.3).

Proof. We consider each part in turn.

(1) Unfolding Notation 10.2.1 we have $(\boldsymbol{\lambda} a.x)\bullet\partial u = (\forall a.(\partial a -\!\!\bullet x))\bullet\partial u$. Renaming using Lemma 4.1.3 if necessary, assume $a\#u$ so that by Theorem 2.3.1 also $a\#\partial u$. By $(\bullet\forall)$ $(\forall a.(\partial a -\!\!\bullet x))\bullet\partial u \leq \forall a.((\partial a -\!\!\bullet x)\bullet\partial u)$. By Lemma 4.3.3 $\forall a.((\partial a -\!\!\bullet x)\bullet\partial u) \leq ((\partial a -\!\!\bullet x)\bullet\partial u)[a{\mapsto}u]$. By Lemma 9.1.9 and $(\sigma\bullet)$ and $(\sigma\#)$, $((\partial a -\!\!\bullet x)\bullet\partial u)[a{\mapsto}u] \leq (\partial u -\!\!\bullet (x[a{\mapsto}u]))\bullet\partial u$. By $(\bullet\epsilon)$ $(\partial u -\!\!\bullet (x[a{\mapsto}u]))\bullet\partial u \leq x[a{\mapsto}u]$.
(2) Suppose $a\#x$. Unfolding Notation 10.2.1 $\boldsymbol{\lambda} a.(x\bullet\partial a) = \forall a.(\partial a -\!\!\bullet (x\bullet\partial a))$. By $(\bullet\eta)$ $x \leq \partial a -\!\!\bullet (x\bullet\partial a)$, so by Lemma 4.1.7 $\forall a.x \leq \forall a.(\partial a -\!\!\bullet (x\bullet\partial a))$. Since $a\#x$, x is its own a-fresh limit, that is, $\forall a.x = x$.

□

REMARK 10.2.5. $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ gives a model of β-reduction and η-expansion. The reverse inclusions do not follow, but they are not forbidden:

— There exist models such that $x[a{\mapsto}u] \not\leq (\lambda a.x)\bullet\partial u$ (so that we do not have β-equality) and $a\#x$ for some x and yet $\lambda a.(x\bullet\partial a) \not\leq x$.

— There also exist models such that $x[a{\mapsto}u] = (\lambda a.x){\bullet}\partial u$ and for all x, $a\#x$ implies $\lambda a.(x{\bullet}\partial a) = x$. To obtain one, choose any λ-equality theory Π (Definition 10.3.7; $\beta\eta$-equality would do here) and construct $points_{\Pi}$ from Definition 11.1.3 and the subsequent constructions.

10.2.2. λ as a sets operation in $F(\mathfrak{D})$. We take a moment to perform a sanity check by examining $\boldsymbol{\lambda}a$ (Notation 10.2.1) for the specific case of the sets representation $\mathfrak{T} = F(\mathfrak{D})$ (Definition 9.5.1) of $\mathfrak{D} \in \mathsf{inDi}\forall_{\bullet}$.

PROPOSITION 10.2.6. *Suppose $\mathfrak{D} \in \mathsf{inDi}\forall_{\bullet}$, $X \in opens(F(\mathfrak{D}))$, and $u \in |F(\mathfrak{D})^{\partial}|{=}|\mathfrak{D}^{\partial}|$, and let p range over elements of $|F(\mathfrak{D})|$, which are prime filters in $\mathfrak{D}$. Then for every $u \in |\mathfrak{D}^{\partial}|$,*

$$p \in \boldsymbol{\lambda}a.X \quad \textit{implies} \quad p{\circ}\partial u \subseteq X[a{\mapsto}u].$$

Proof. Recall the unpacking of $\boldsymbol{\lambda}a$ from Remark 10.2.2. We reason as follows:

$$\begin{array}{lll}
p \in \boldsymbol{\lambda}a.X & \Leftrightarrow p \in \forall a.(\partial a{-\!\bullet}X) & \text{Notation 10.2.1} \\
& \Leftrightarrow \forall u{\in}|F(\mathfrak{D})^{\partial}|.p \in (\partial a{-\!\bullet}X)[a{\mapsto}u] & \text{Lemma 4.3.3} \\
& \Leftrightarrow \forall u{\in}|\mathfrak{D}^{\partial}|.p \in (\partial a{-\!\bullet}X)[a{\mapsto}u] & F(\mathfrak{D})^{\partial}{=}\mathfrak{D}^{\partial} \text{ by Thm 7.5.4} \\
& \Rightarrow \forall u{\in}|\mathfrak{D}^{\partial}|.p \in (\partial a)[a{\mapsto}u]{-\!\bullet}X[a{\mapsto}u] & \text{Lem 9.1.9, Prop 9.6.5, Thm 9.4.11} \\
& \Leftrightarrow \forall u{\in}|\mathfrak{D}^{\partial}|.p \in \partial u{-\!\bullet}X[a{\mapsto}u] & (\sigma\mathbf{a}),\ \partial \text{ morphism (Def 4.4.4)} \\
& \Leftrightarrow \forall u{\in}|\mathfrak{D}^{\partial}|.p{\circ}\partial u \subseteq X[a{\mapsto}u] & \text{Proposition 9.2.6}
\end{array}$$

□

REMARK 10.2.7. Continuing the notation of Proposition 10.2.6, one might expect $p \in \boldsymbol{\lambda}a.X$ to also be *equivalent* to $\forall u{\in}|\mathfrak{D}^{\partial}|.p{\circ}\partial u \subseteq X[a{\mapsto}u]$. This seems to not be the case, because Lemma 9.1.9 used in the proof above is an inequality and not an equality.

10.3. Idioms

It is convenient to generalise λ-syntax a little. Recall from Definition 3.1.1 that a *termlike σ-algebra* expresses in nominal algebra the property of 'having a substitution action over itself'.

DEFINITION 10.3.1. A **(λ-)idiom** is a termlike σ-algebra $\mho$ equipped with two further equivariant functions

$$\begin{array}{ll}
\bullet_{\mho} : \mho \times \mho \to \mho & \text{and} \\
\lambda_{\mho} : \mathbb{A} \times \mho \to \mho &
\end{array}$$

such that for all $a \in \mathbb{A}$ and $x, y \in |\mho|$ and $u \in |\mho|$:

(1) $a\#\lambda_{\mho}a.x$ (this justifies quantifier notation: λ abstracts the atoms argument a).
(2) If $b\#u$ then $(\lambda_{\mho}b.x)[a{:=}u]_{\mho} = \lambda_{\mho}b.(x[a{:=}u]_{\mho})$.
(3) $(x{\bullet}_{\mho}y)[a{:=}u]_{\mho} = (x[a{:=}u]_{\mho}){\bullet}_{\mho}(y[a{:=}u]_{\mho})$.
(4) If $x, y \in |\mho|$ then $x{\bullet}_{\mho}y \in |\mho|$, and $\lambda_{\mho}a.x \in |\mho|$.

Above, we use the fact that because $\mho$ is a termlike σ-algebra, it interprets atoms as $\mathsf{atm}_{\mho}(a)$ and as a σ-action $x[a{:=}u]_{\mho}$.

REMARK 10.3.2. The canonical example of a λ-idiom is LmTm; λ-terms up to α-equivalence, with their natural substitution, application, and λ-actions.

NOTATION 10.3.3.— We write $a_{\mho}$ for $\mathsf{atm}_{\mho}(a)$, or just a.
— We write $x[a{:=}u]$ for $x[a{:=}u]_{\mho}$.
— We write xy for $x{\bullet}_{\mho}y$.
— We write $\lambda a.x$ for $\lambda_{\mho}a.x$.

The conditions of Definition 10.3.1 can be written in this notation, and using Corollary 2.1.10, thus:

$$\begin{aligned} b\#x \Rightarrow \quad\quad\quad \lambda a.x &= \lambda b.(b\ a)\cdot x \\ b\#u \Rightarrow (\lambda b.x)[a{:=}u] &= \lambda a.(x[a{:=}u]) \\ (xy)[a{:=}u] &= x[a{:=}u]\, y[a{:=}u] \end{aligned}$$

Note that it follows already from axiom $(\sigma\mathbf{a})$ in Figure 1 that $a_{\mho}[a{:=}u] = u$.

Notation 10.3.4. In what follows, what the variables, substitution, application, and λ of an idiom $\mho$ have to be, will always be clear. For the rest of this section fix some λ-idiom $\mho$.

Notation 10.3.5. We call elements of $|\mho|$ **phrases**. We let s and t range over phrases in $|\mho|$, and also u and v range over phrases in $|\mho|$.

Remark 10.3.6. Phrases of a λ-idiom 'look like' terms of λ-syntax up to α-equivalence, inasmuch as they must support variables, substitution, a binary operator which we suggestively call application, and a variable-abstractor which we suggestively call λ. However, we do not insist that phrases *be* λ-terms; they need not even be syntax. They just have to support nominal algebraic models of variables, substitution, application and a λ-abstraction. Nothing about the constructions that follow immediately below depends on $\mho$ being syntactic.

Definition 10.3.7. Suppose $\mho$ is a λ-idiom. Call an equivariant preorder $\mathcal{R}$ on phrases **compatible** when for all $s, s', t, t' \in |\mho|$ and $u \in |\mho|$:[34]

(1) If $s\ \mathcal{R}\ s'$ and $t\ \mathcal{R}\ t'$ then $st\ \mathcal{R}\ s't'$.
(2) If $s\ \mathcal{R}\ s'$ then $\lambda a.s\ \mathcal{R}\ \lambda a.s'$.
(3) If $s\ \mathcal{R}\ s'$ then $s[a{:=}u]\ \mathcal{R}\ s'[a{:=}u]$.
(4) $(\lambda a.s)u\ \mathcal{R}\ s[a{:=}u]$. This is β**-reduction**.
(5) If a is not free in s then $s\ \mathcal{R}\ \lambda a.(sa)$. This is η**-expansion**.

A λ**-reduction theory** is a compatible preorder on an idiom $\mho$, and a λ**-equality theory** is a compatible equivalence relation on $\mho$.[35] Π will range over λ-reduction theories.

Notation 10.3.8. If Π is a λ-reduction theory then we may write $s {\to_\Pi} t$ or $\Pi \vdash s{\to}t$ or $(s{\to}t) \in \Pi$ for $s\ \Pi\ t$.

If Π is a λ-equality theory, so that Π is an equivalence relation, then we may write $s =_\Pi t$ or $\Pi \vdash s = t$ or $(s = t) \in \Pi$ for $s\ \Pi\ t$.

10.4. A sound denotation for the λ-calculus

Any $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ has the structure of $\forall$, $\bullet$, and $-\!\bullet$, so we can immediately interpret the λ-calculus in $\mathcal{D}$. Lo and behold, the interpretation is sound. This is Definition 10.4.1 and Theorem 10.4.7.

The denotation we obtain is *absolute*, meaning that a variable/atom a is interpreted 'as itself'—there is no valuation. Slightly more formally, a denotation is absolute when variable symbols in the syntax map to fixed entities in the denotation. In the case of this paper, a (more strictly: $a_\mho$) is interpreted as ∂a (more strictly: $\partial_{\mathcal{D}} a_{\mathcal{D}^\partial}$, see Notation 4.5.3).

The role of a valuation is played by the σ-action. If we have some $x \in |\mathcal{D}|$ and want to 'evaluate' any a in it to become u, then we just apply $[a{\mapsto}u]$. This nominal approach to valuations using σ-algebras is *more general* than the usual Tarski denotation based on valuations; to see why, see the discussion in [Gab12, Remark 8.18].

[34] $\mathcal{R}$ being a preorder means precisely that it is transitive and reflexive. (A partial order is an antisymmetric preorder.)
Equivariance means that $s\ \mathcal{R}\ s'$ implies $\pi{\cdot}s\ \mathcal{R}\ \pi{\cdot}s'$. See Definition 2.1.6 and the discussion in Subsection 2.2.2. Another description of this is that $\mathcal{R}$ as an element of $pow(\mathsf{LmTm} \times \mathsf{LmTm})$ (Subsection 2.4.1), has $supp(\mathcal{R}) = \varnothing$.

[35] An equivalence relation is a symmetric preorder, so a λ-equality theory is, as one would expect, just a symmetric λ-reduction theory.

Definition 10.4.1. Suppose $\mathcal{D} \in \mathsf{inDi}\forall_{\bullet}$. Define a **denotation** of λ-terms by:[36]

$$\begin{aligned} [\![a]\!]^{\mathcal{D}} &= \partial_{\mathcal{D}} a_{\mathcal{D}^{\partial}} \\ [\![\lambda a.s]\!]^{\mathcal{D}} &= \boldsymbol{\lambda} a.[\![s]\!]^{\mathcal{D}} \\ [\![s's]\!]^{\mathcal{D}} &= [\![s']\!]^{\mathcal{D}} \bullet [\![s]\!]^{\mathcal{D}} \end{aligned}$$

— Write $\mathcal{D} \vDash s \leq t$ when $[\![s]\!]^{\mathcal{D}} \leq [\![t]\!]^{\mathcal{D}}$.
— Write $\mathcal{D} \vDash \Pi$ when $\mathcal{D} \vDash s \leq t$ for every $(s \to t) \in \Pi$.
— Write $\Pi \vDash s \leq t$ when $\forall \mathcal{D} {\in} \mathsf{inDi}\forall_{\bullet}.\big(\mathcal{D} \vDash \Pi \Rightarrow \mathcal{D} \vDash s \leq t\big)$.

Remark 10.4.2. Suppose $\mathcal{D} \in \mathsf{inDi}\forall_{\bullet}$. Recall from Definition 4.4.3 that $\mathcal{D}^{\partial}$ is the termlike σ-algebra over which substitution in $\mathcal{D}$ is defined, and recall that (since $\mathcal{D} \in \mathsf{inDi}\forall_{\bullet}$ is *impredicative*; see Definition 4.5.1) we assume a σ-algebra morphism $\partial_{\mathcal{D}}$ from $\mathcal{D}^{\partial}$ to $\mathcal{D}$.

Recall from Notation 4.5.3 that we write $\partial\mathcal{D}$ for the sets image of $\partial_{\mathcal{D}}$, i.e. $\partial\mathcal{D} = \{\partial_{\mathcal{D}} u \mid u \in \mathcal{D}^{\partial}\} \subseteq |\mathcal{D}|$, and recall that we call this image the **programs** of $\mathcal{D}$.

Definition 10.4.3. Call $\mathcal{D} \in \mathsf{inDi}\forall_{\bullet}$ **replete** if $\partial\mathcal{D}$ is closed under application and λ. That is:

— If $x, y \in \partial\mathcal{D}$ then $x \bullet y \in \partial\mathcal{D}$.
— If $x \in \partial\mathcal{D}$ then $\boldsymbol{\lambda} a.x \in \partial\mathcal{D}$.

Remark 10.4.4. Note that $\partial a \in \partial\mathcal{D}$ is a fact, where ∂a is shorthand for $\partial_{\mathcal{D}} a_{\mathcal{D}^{\partial}}$. If $\mathcal{D}$ is replete then programs are closed under taking variables, application, or λ-abstraction, and intuitively this tells us the following:

> If $\mathcal{D}$ is replete then its programs include denotations for all λ-terms.

This intuition is exactly the notion of repleteness used in [GG10] (we called it *faithful* there, but that terminology clashes with faithfulness of functors in category theory). In this paper we are using nominal techniques, so we can give a name-based semantic treatment of λ, so that Definition 10.4.3 can be more abstract than it needed to be in [GG10], and it needs make no explicit mention of λ-term syntax.

Remark 10.4.5. Definition 10.4.3 is needed for Lemma 10.4.6. In any case, we are most interested in $\mathcal{D}$ that are replete, since we are interested in models of the λ-calculus and we would expect λ-terms to denote programs.

So if $\mathcal{D}$ is replete then Definition 10.4.1 generates programs, which can be substituted for in $\mathcal{D}$, and we can express Lemma 10.4.6:

Lemma 10.4.6. *Suppose $\mathcal{D} \in \mathsf{inDi}\forall_{\bullet}$ is replete. Then $[\![s]\!]^{\mathcal{D}}[a{\mapsto}[\![u]\!]^{\mathcal{D}}] = [\![s[a{:=}u]]\!]^{\mathcal{D}}$.*

($[\![u]\!]^{\mathcal{D}}$ always exists, but repleteness ensures that $[\![u]\!]^{\mathcal{D}} \in \partial\mathcal{D}$ so that the substitution $[a{\mapsto}[\![u]\!]^{\mathcal{D}}]$ also exists.)

Proof. By induction on s.

— *The case of a.* By Definition 10.4.1 $[\![a]\!]^{\mathcal{D}} = \partial_{\mathcal{D}} a_{\mathcal{D}^{\partial}}$. We use Lemma 4.5.5.

[36] In the case for $[\![a]\!]^{\mathcal{D}}$, $a_{\mathcal{D}^{\partial}}$ is the copy of a in the termlike σ-algebra $\mathcal{D}^{\partial}$ (Definition 3.1.1) and $\partial_{\mathcal{D}}$ is the function mapping $\mathcal{D}^{\partial}$ to $\mathcal{D}$ (see Definition 4.5.1 and Notation 4.5.3). We have written $\partial_{\mathcal{D}} a_{\mathcal{D}^{\partial}}$ as just ∂a, but here we prefer the more careful notation.

Of course, if we wanted to be *really* careful we would also mention that $a_{\mathcal{D}^{\partial}}$ is itself shorthand for $\mathtt{atm}_{\mathcal{D}^{\partial}}(a)$ from Definition 3.1.1. But the reader probably is not interested in that high level of pedantry, and may even be confused by it, so we will not labour the point further.

— *The case of $\lambda b.s$.* Renaming if necessary assume $b\#u$. We reason as follows:

$$\begin{aligned}
[\![(\lambda b.s)[a{:=}u]]\!]^{\mathcal{D}} &= [\![\lambda b.(s[a{:=}u])]\!]^{\mathcal{D}} && \text{Fact of } \lambda\text{-terms, } b\#u \\
&= \boldsymbol{\lambda} b.[\![s[a{:=}u]]\!]^{\mathcal{D}} && \text{Definition 10.4.1} \\
&= \boldsymbol{\lambda} b.([\![s]\!]^{\mathcal{D}}[a{\mapsto}[\![u]\!]^{\mathcal{D}}]) && \text{ind. hyp.} \\
&= (\boldsymbol{\lambda} b.[\![s]\!]^{\mathcal{D}})[a{\mapsto}[\![u]\!]^{\mathcal{D}}] && \text{Lemma 10.2.3} \\
&= [\![\lambda b.s]\!]^{\mathcal{D}}[a{\mapsto}[\![u]\!]^{\mathcal{D}}] && \text{Definition 10.4.1}
\end{aligned}$$

In the use of Lemma 10.2.3 above we know $b\#[\![u]\!]^{\mathcal{D}}$ by Theorem 2.3.1 since $b\#u$.

— *The case of $s's$.* Routine using the inductive hypothesis and $(\sigma\bullet)$ from Definition 9.1.1.

□

Recall the notations $s{\to}_{\Pi}t$ and $\Pi \vdash s \to t$ from Notation 10.3.8, applied here to the idiom LmTm (λ-terms). Recall the notation $\Pi \vDash s \leq t$ from Definition 10.4.1.

THEOREM 10.4.7 (Soundness). *Suppose $\mathcal{D} \in \mathsf{inDi}\forall_{\bullet}$ is replete. Then if $s{\to}_{\Pi}t$ and $\mathcal{D} \vDash \Pi$ then $[\![s]\!]^{\mathcal{D}} \leq [\![t]\!]^{\mathcal{D}}$.*
In other words, $\Pi \vdash s \to t$ implies $\Pi \vDash s \leq t$.

(The reverse implication also holds; see Theorem 11.9.5.)

Proof. We consider the rules defining a compatible relation on λ-terms (Definition 10.3.7):

(1) *If $s\ \mathcal{R}\ s'$ and $u\ \mathcal{R}\ u'$ then $su\ \mathcal{R}\ s'u'$.* We use Lemma 9.1.7.
(2) *If $s\ \mathcal{R}\ s'$ then $\lambda a.s\ \mathcal{R}\ \lambda a.s'$.* We use Lemmas 4.1.7 and 9.1.7.
(3) *If $s\ \mathcal{R}\ s'$ then $s[a{:=}u]\ \mathcal{R}\ s'[a{:=}u]$.* We use Lemmas 10.4.6 and 4.3.2.
(4) $(\lambda a.s)t\ \mathcal{R}\ s[a{:=}t]$. We use Lemma 10.4.6 and part 1 of Proposition 10.2.4.
(5) *If a is not free in s then $s\ \mathcal{R}\ \lambda a.(sa)$.* If $a\#s$ then by Theorem 2.3.1 also $a\#[\![s]\!]^{\mathcal{D}}$. We use part 2 of Proposition 10.2.4.

□

10.5. Interlude: axiomatising the λ-calculus in nominal algebra

Some words on where we are and where we are going.

Nominal algebra considers equality over nominal sets.[37] It was introduced in two papers [GM06a; GM06b] where it was applied to axiomatise to substitution and first-order logic respectively.[38] Both applications feature α-equivalence and freshness side-conditions, which are of course just what nominal sets were developed to model, so this was very natural.

See [GP01] or see [Gab11a; Gab13] for surveys.

In [GM08b; GM10] nominal algebra was applied to the λ-calculus, extending an incomplete axiomatisation from [FG07]—Henkin style models of such axioms were considered in [GM11] and found to have some interesting properties. In particular the axiomatisation is sound and complete—so the axioms below *really do* axiomatise the λ-calculus; and this proof, in greatly strengthened form, has become the duality, soundness, and completeness results of the current paper.

So an axiomatisation of the λ-calculus is implicit in this paper. The reader could extract it by tracing through Notation 10.2.1 and the axioms of $\mathsf{inDi}\forall_{\bullet}$. We do not have to write out this theory to prove soundness in this paper, because the notion of λ-calculus we use in this paper is the standard one based on λ-term syntax and reduction.

Yet the axiomatisation is there in the background, and for the reader's convenience it might be illuminating to write it out.

Consider a nominal set $\mathcal{X}$.

[37] It is descended from nominal rewriting, which considers rewriting over nominal terms [FGM04; FG07].

[38] The papers wrote axioms and proved them sound and complete. So we really did check that the axioms do what one would expect them to do; no more and no less.

We assume equivariant functions $\mathsf{atm} : \mathbb{A} \to \mathcal{X}$ and $\mathsf{sub} : \mathcal{X} \times \mathbb{A} \times \mathcal{X} \to \mathcal{X}$ and impose the axioms of a termlike σ-algebra from Figure 1:

$$
\begin{array}{lrl}
(\sigma\mathbf{a}) & & a[a\mapsto x] = x \\
(\sigma\mathbf{id}) & & x[a\mapsto a] = x \\
(\sigma\#) & a\#x \Rightarrow & x[a\mapsto u] = x \\
(\sigma\alpha) & b\#x \Rightarrow & x[a\mapsto u] = ((b\ a)\cdot x)[b\mapsto u] \\
(\sigma\sigma) & a\#v \Rightarrow & x[a\mapsto u][b\mapsto v] = x[b\mapsto v][a\mapsto u[b\mapsto v]]
\end{array}
$$

Here we sugar $\mathsf{atm}(a)$ to just a and $\mathsf{sub}(x, a, u)$ to just $x[a\mapsto u]$.

Next we assume equivariant functions $\mathsf{app} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $\mathsf{lam} : \mathbb{A} \times \mathcal{X} \to \mathcal{X}$, and impose the axioms of β- and η-equality:

$$
\begin{array}{lrl}
(\lambda\alpha) & b\#x \Rightarrow & \boldsymbol{\lambda} a.x = \boldsymbol{\lambda} b.(b\ a)\cdot x \\
(\beta{=}) & & (\boldsymbol{\lambda} a.x)y = x[a\mapsto y] \\
(\eta{=}) & a\#x \Rightarrow & \boldsymbol{\lambda} a.(xa) = x
\end{array}
$$

Here we sugar $\mathsf{app}(x, y)$ to xy and $\mathsf{lam}(a, x)$ to $\boldsymbol{\lambda} a.x$.

A few notes on this axiomatisation:

— The axiomatisation of [GM08b; GM10] identified substitution with a β-reduct. The axiomatisation above distinguishes substitution and β-reducts. This turns out to be important for making the results in this paper work; for more discussion see the Conclusions.
— The body of this paper is based on lattices, so we do not *assume* β- or η-equality; we only assume β-reduction and η-expansion. (The equalities might happen to be valid anyway, see for instance λ-equality theories in Definition 10.3.7.) This is also important for this paper.

In summary the axiomatisation above is a special case of a generalisation of [GM08b; GM10], which is itself a complete extension of a rewrite theory from [FG07; GM06a].

The axiomatisation above is also what we are aiming for, and models of λ-equality theories constructed in Section 11 are models of the axioms above, though the demands of our main results are such that we do not phrase matters in that specific form.

11. REPRESENTATION OF THE λ-CALCULUS IN INSPECT$\forall_\bullet$

In Section 10 we showed how any $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ / (dually) any $\mathcal{T} \in \mathsf{inSpect}\forall_\bullet$, gives a sound abstract / (dually) concrete interpretation of the untyped λ-calculus.

The next step is to prove completeness. This is Theorem 11.9.5. The method is to construct a nominal spectral space points_Π out of a λ-reduction theory Π, in which only those subset inclusions are valid that are insisted on by Π.

points_Π is a rich structure. Notable technical definitions and results are Definition 11.1.1 and Proposition 11.1.6 ($a \natural_\sigma p$ and its equivalence with $a\#p$), completeness under finitely-supported sets unions and intersections (Proposition 11.1.8), the σ-action on points (Definition 11.4.1) and its two characterisation in Subsection 11.4.2—one in terms of the now-ubiquitous $\reflectbox{$\mathsf{N}$}$.

For this section, fix the following data:

— Fix a λ-idiom $\mho$ (Definition 10.3.1).
— Fix a λ-reduction theory Π on $\mho$ (Definition 10.3.7).

s, s', s'', and t will range over elements of $|\mho|$, and u and v will range over elements of $|\mho|$.

11.1. Π-points and σ-freshness

Given a subset $p \subseteq |\mho|$, we can suggest two notions of 'a is fresh for p':

— One inherited from nominal techniques: $\reflectbox{$\mathsf{N}$} b.(b\ a)\cdot p = p$. We write this $a\#p$.
— One inherited from our syntactic intuitions: if $s \in p$ then $\forall u{\in}|\mho|.s[a{:=}u] \in p$. We will make this formal in Definition 11.1.1 and write it $a \natural_\sigma p$.

A *point* is then defined to be a set of phrases for which these two notions of freshness coincide. This is Definition 11.1.3. The interesting part is condition 2, which is not obviously just "$a\#p \Leftrightarrow a \sharp_\sigma p$"—for this, see the important and beautiful Proposition 11.1.6, which proves this from surprisingly little in the way of assumptions.

We conclude with Proposition 11.1.8, an important result asserting that $points_\Pi$ is complete for finitely-supported diagrams (i.e. finitely-supported sets of points have an intersection that is also a point). This has some very useful consequences: for instance it makes possible the use of $\bigcap$ in Definitions 11.3.1, and 11.4.1, and also in Corollary 11.4.7.

Definition 11.1.1. Suppose $p \subseteq |\mho|$. Define $a \sharp_\sigma p$ by:

$$a \sharp_\sigma p \quad \text{when} \quad \forall s{\in}|\mho|.\forall u{\in}|\mho|.(s \in p \Rightarrow s[a{:=}u] \in p)$$

If $a \sharp_\sigma p$ we say that a is **σ-fresh** for p.

Remark 11.1.2. We rewrite Definition 11.1.1 twice:

— $a \sharp_\sigma p$ when $\forall s{\in}|\mho|.(s \in p \Rightarrow \forall u{\in}|\mho|.s[a{:=}u] \in p)$.
— $a \sharp_\sigma p$ when $\forall u{\in}|\mho|.p \subseteq p[u{\leftarrow\!\!\dashv}a]$.

$p[u{\leftarrow\!\!\dashv}a]$ is from Definition 3.3.1; but see also Definition 11.2.3 and Lemma 11.2.6 below.

Recall the notion of a λ-*reduction theory* from Definition 10.3.7.

Definition 11.1.3. Suppose Π is a λ-reduction theory. Call a subset $p \subseteq |\mho|$ a Π**-point** when:

(1) $\forall s,t{\in}|\mho|.(s \in p \wedge s{\to}_\Pi t) \Rightarrow t \in p$. We call p **closed under** Π.
(2) $\mathsf{N}a.a \sharp_\sigma p$. We call p **finitely σ-supported**.

Write $|points_\Pi|$ for the set of Π-points.

Remark 11.1.4. Note that $\varnothing$ (the empty set of phrases) is a point. This will be useful in Proposition 11.8.3 to prove that the set of all points is compact and covered by $\{\varnothing^\bullet\}$ (the $\text{-}^\bullet$ notation will be defined in Definition 11.4.1).

Lemma 11.1.5. *Suppose $p \subseteq |\mho|$. Then if $a \sharp_\sigma p$ then $a\#p$.*

Proof. Suppose $a \sharp_\sigma p$. To show $a\#p$ it suffices by Corollary 2.1.10 to show that for fresh b (so $b\#p$) $(b\ a){\cdot}p = p$. The permutation action is pointwise (Definition 2.2.1) so it suffices to show that for any $s \in |\mho|$, $s \in p$ implies $(b\ a){\cdot}s \in p$.

By Lemma 3.2.6 $(b\ a){\cdot}s = s[a{:=}c][b{:=}a][c{:=}b]$ for fresh c (so $c\#p$ and c is distinct from a and b). Now $s \in p$ and $a\#p$ so $s[a{:=}c] \in p$. Also $b\#p$ so $s[a{:=}c][b{:=}a] \in p$. Also $c\#p$ so $s[a{:=}c][b{:=}a][c{:=}b] \in p$, and we are done. □

We discussed at the introduction to this subsection why Proposition 11.1.6 is interesting:

Proposition 11.1.6. *Suppose $p \in |points_\Pi|$. Then*

$$a\#p \quad \textit{if and only if} \quad a \sharp_\sigma p.$$

Proof. First, suppose $a\#p$. By condition 2 of Definition 11.1.3 and Theorem 2.3.9 we have $a \sharp_\sigma p$.

Conversely, if $a \sharp_\sigma p$ then by Lemma 11.1.5 $a\#p$. □

Corollary 11.1.7. *$p \in |points_\Pi|$ is finitely supported in the sense of Definition 2.1.4 and of Subsection 2.4.1.*

Proof. Direct from Proposition 11.1.6 and condition 2 of Definition 11.1.3. □

We conclude the subsection with Proposition 11.1.8, a useful result with an attractive proof:

Proposition 11.1.8. *Suppose $\mathcal{P} \subseteq |points_\Pi|$ is finitely supported. Then*

$$\bigcap \mathcal{P} \in points_\Pi \quad \textit{and} \quad \bigcup \mathcal{P} \in points_\Pi .$$

In words: finitely supported intersections and union of points, are points.

Proof. We check the conditions of Definition 11.1.3 for $\bigcap \mathcal{P} \in points_\Pi$. Condition 1 is by a routine calculation.

For condition 2, suppose a is fresh (so $a\#\mathcal{P}$). To prove $a \,\sharp_\sigma \bigcap \mathcal{P}$ we need to show that $s \in \bigcap \mathcal{P}$ implies $\forall u{\in}|\mho|.s[a{:=}u]{\in}\bigcap \mathcal{P}$.

Consider $s \in \bigcap \mathcal{P}$ and any $p \in \mathcal{P}$, so that $s \in p$. Suppose $a\#\mathcal{P}$. We want to prove $\forall u{\in}|\mho|.s[a{:=}u]{\in}p$. We cannot do this directly since we do not necessarily know that $a\#p$ (cf. Lemma 2.4.1).

So choose fresh c (so $c\#\mathcal{P}, p, s$). Since $p \in \mathcal{P}$ also $(c\ a){\cdot}p \in (c\ a){\cdot}\mathcal{P} \stackrel{\text{Cor 2.1.10}}{=} \mathcal{P}$. So $s \in (c\ a){\cdot}p$ and therefore $(c\ a){\cdot}s \in p$. By assumption $c\#p$ so by Proposition 11.1.6 $c \,\sharp_\sigma\, p$ so $\forall u.((c\ a){\cdot}s)[c{:=}u] \in p$. We α-convert, and conclude that $\forall u{\in}|\mho|.s[a{:=}u] \in p$ as required.

The reasoning for $\bigcup \mathcal{P} \in points_\Pi$ is almost identical. □

11.2. Constructing Π-points, and their ɒ-algebra structure

Recall the notion of idiom $\mho$ from Definition 10.3.1, the notion of λ-reduction theory Π from Definition 10.3.7, and the notion of point p from Definition 11.1.3.

Definition 11.2.1. Suppose $s \in |\mho|$. Define $s{\uparrow}_\Pi$ by

$$s{\uparrow}_\Pi = \{s' \in |\mho| \mid s {\to}_\Pi s'\}.$$

Lemma 11.2.2. *If $s \in |\mho|$ then $s{\uparrow}_\Pi$ is a point.*

Proof. We consider the conditions of Definition 11.1.3. By transitivity if $s'' \in s{\uparrow}_\Pi$, meaning $s{\to}_\Pi s''$, and $s''{\to}_\Pi s'$ then $s{\to}_\Pi s'$.

Suppose $a\#s$ and suppose $s' \in s{\uparrow}_\Pi$. By condition 3 of Definition 10.3.7 also $s[a{:=}u]{\to}_\Pi s'[a{:=}u]$ and by $(\sigma\#)$ $s[a{:=}u] = s$. It follows that $s'[a{:=}u] \in s{\uparrow}_\Pi$ for every u. Thus $s{\uparrow}_\Pi$ is finitely σ-supported. □

Definition 11.2.3. Give $p \subseteq |\mho|$ a permutation action and an ɒ-action following Definitions 3.3.1 and 7.2.1:

$$\pi{\cdot}p = \{\pi{\cdot}r \mid r \in p\} \qquad p[u{\mapsfrom}a] = \{s \mid s[a{:=}u] \in p\} \quad (u \in |\mho|)$$

Write $points_\Pi$ for (what will prove will be) the ɒ-algebra with underlying set $|points_\Pi|$ and the permutation and ɒ-actions defined above.

Our notation suggests that $[u{\mapsfrom}a]$ is an ɒ-action. This is true, but we must prove it: this is Lemma 11.2.4 and Corollary 11.2.5.

Lemma 11.2.4. *If $p \in |points_\Pi|$ and $u \in |\mho|$ then $p[u{\mapsfrom}a]$ is a Π-point.*

Proof. We check the conditions of Definition 11.1.3, freely using Proposition 3.3.2:

(1) Suppose $s[a{:=}u] \in p$ and $s \to s' \in \Pi$. By condition 3 of Definition 10.3.7 $s[a{:=}u]{\to}_\Pi s'[a{:=}u]$. By assumption p is closed under Π, and so $s'[a{:=}u] \in p$.

(2) Suppose b is fresh (so $b\#p, a, u$) and consider $s[a{:=}u] \in p$. By assumption p is finitely σ-supported and $b\#p$ so by Proposition 11.1.6 $b \,\sharp_\sigma\, p$. Therefore

$$\forall v.s[a{:=}u][b{:=}v] \in p$$

Now $b\#u$, so by $(\sigma\sigma)$ for any $v' \in |\mho|$ we have $s[b{:=}v'][a{:=}u] = s[a{:=}u][b{:=}v'[a{:=}u]]$. It follows (taking all v of the form $v'[a{:=}u]$ above) that $\forall v.s[b{:=}v][a{:=}u] \in p$, and so $\forall v.s[b{:=}v] \in p[u{\mapsfrom}a]$ as required.

□

Corollary 11.2.5. *$points_\Pi$ is indeed an $\mho$-algebra.*

Proof. Lemma 11.2.4 proves that $[u\hookleftarrow a]$ maps points to points. We use Proposition 3.3.4. □

We conclude with a technical result which will be useful for Lemma 11.5.1:

Lemma 11.2.6. *If $p \in |points_\Pi|$ and $a\#p$ and $u \in |\mho|$ then $p \subseteq p[u\hookleftarrow a]$.*

Proof. Suppose $a\#p$. By Proposition 11.1.6 $a \,\sharp_\sigma\, p$, so by Definition 11.1.1 $s \in p$ implies $s[a{:=}u] \in p$. The result follows by Proposition 3.3.2. □

11.3. Some further operations on points

11.3.1. The operations: $\wedge$, $\forall$, $\bullet$, and $-\!\!\bullet$ on points. Fix some λ-reduction theory Π.

Definition 11.3.1. Suppose $p, q \subseteq |\mho|$. Define the following operations:

$$\begin{aligned} p\wedge q &= p \cup q \\ \forall a.p &= \bigcap\{r \in |points_\Pi| \mid p \subseteq r \wedge a\#r\} \\ p\bullet q &= \bigcup\{(st){\Uparrow}_\Pi \mid s \in p,\ t \in q\} \\ q-\!\!\bullet p &= \bigcap\{r \in |points_\Pi| \mid p \subseteq r\bullet q\} \end{aligned}$$

Remark 11.3.2. Two things about Definition 11.3.1 might seem odd:

— $p\wedge q$ is a sets *union* (not a sets intersection). This is a contravariance typical in duality results. See Proposition 11.7.11 for the treatment of $\wedge$, and see Lemma 11.7.6 for a clearer view of the contravariance in this case.
— There is no $p\vee q$, *even though* in Proposition 11.1.8 we proved that a finite sets intersection of points is a point. This is because sets intersection of points does not interact correctly with the σ-action, see Remark 11.5.8. For that, we need to consider sets of points; see Corollary 11.7.9.

Lemma 11.3.3. *If p and q are Π-points then so are $p\wedge q$, $\forall a.p$, $p\bullet q$, and $q-\!\!\bullet p$.*

Proof. All from Proposition 11.1.8, and for $p\bullet q$ also Lemma 11.2.2. □

Lemma 11.3.4. *Suppose $p, q \subseteq |\mho|$ and suppose $r \in |points_\Pi|$. Then:*

(1) $q-\!\!\bullet p \subseteq r$ if and only if $p \subseteq r\bullet q$.
(2) If $p, q \in |points_\Pi|$ then $p \subseteq (q-\!\!\bullet p)\bullet q$ and $q-\!\!\bullet(p\bullet q) \subseteq p$.
(3) If $p \subseteq p'$ then $q-\!\!\bullet p \subseteq q-\!\!\bullet p'$.

Proof. Part 1 is from Definition 11.3.1. Part 2 follows using Lemma 11.3.3 since $q-\!\!\bullet p \subseteq q-\!\!\bullet p$ and $p\bullet q \subseteq p\bullet q$. For part 3 we note by part 2 that $p' \subseteq (q-\!\!\bullet p')\bullet q$, deduce that $p \subseteq (q-\!\!\bullet p')\bullet q$, and use part 1 to conclude that $q-\!\!\bullet p \subseteq q-\!\!\bullet p'$. □

Lemma 11.3.5. *Suppose $p, q \in points_\Pi$. Then:*

(1) $p \subseteq \forall a.p$.
(2) If $a\#p$ then $p = \forall a.p$.
(3) If $p \subseteq q$ then $\forall a.p \subseteq \forall a.q$.
(4) If $a\#q$ then $(\forall a.p) \subseteq q$ if and only if $p \subseteq q$.

Proof. The first three parts follow by construction in Definition 11.3.1. For the final part, if $\forall a.p \subseteq q$ then $p \subseteq q$ using part 1 of this result, and if $p \subseteq q$ then $\forall a.p \overset{\text{pt }3}{\subseteq} \forall a.q \overset{\text{pt }2}{=} q$. □

11.3.2. $\multimap$ and $\forall$ make λ

Lemma 11.3.6. $q\multimap p = \{s' \in |\mho| \mid \forall r.(p \subseteq r\bullet q \Rightarrow s' \in r)\}$.
As a corollary, $s' \in t{\uparrow_\Pi}\multimap s{\uparrow_\Pi}$ if and only if $s't{\to_\Pi}s$.

Proof. The first part just unpacks Definition 11.3.1.
Now suppose $s' \in t{\uparrow_\Pi}\multimap s{\uparrow_\Pi}$. By the first part, there exists some t' with $t{\to_\Pi}t'$ and $s't'{\to_\Pi}s$. By condition 1 of Definition 10.3.7 $s't{\to_\Pi}s't'$. It follows that $s't{\to_\Pi}s$.
Conversely suppose $s't{\to_\Pi}s$ and choose any s'' with $s{\to_\Pi}s''$. It follows that $s't{\to_\Pi}s''$ and since $t \in t{\uparrow_\Pi}$, we are done. $\square$

Remark 11.3.7. For the reader's convenience we apply Lemma 11.3.6 to some concrete cases. Suppose $\mho = \mathsf{LmTm}$.

— Take $q = a{\uparrow_\Pi} = p$. Then $s' \in a{\uparrow_\Pi}\multimap a{\uparrow_\Pi}$ if and only if $s'a{\to_\Pi}a$. We can calculate that $\lambda a.a \in q\multimap p$ and also $\lambda b.a \in q\multimap p$.
— Assume some implementation of ordered pairs (s,t) and π_1 and π_2 for first and second projection, and take $q = (a,b){\uparrow_\Pi}$ and $p = a{\uparrow_\Pi}$. Then $s' \in (a,b){\uparrow_\Pi}\multimap a{\uparrow_\Pi}$ if and only if $s'(a,b){\to_\Pi}a$. We can calculate that $\lambda b.a \in q\multimap p$ and $\pi_1 \in q\multimap p$.

So we can think of $\multimap$ as a kind of pattern-matching. We refine this to model λ in Proposition 11.3.9.
Recall $s{\uparrow_\Pi}$ from Definition 11.2.1, which is a point by Lemma 11.2.2.

Lemma 11.3.8. $s{\uparrow_\Pi}\bullet t{\uparrow_\Pi} = (st){\uparrow_\Pi}$.

Proof. Unpacking Definitions 11.2.1 and 11.3.1, $u \in s{\uparrow_\Pi}\bullet t{\uparrow_\Pi}$ when $s{\to_\Pi}s'$ and $t{\to_\Pi}t'$ and $s't'{\to_\Pi}u$. Also, $u \in (st){\uparrow_\Pi}$ when $st{\to_\Pi}u$. It is a fact that these two conditions are equivalent. $\square$

Proposition 11.3.9 connects $\forall a$ and $\multimap$ on points, with λa on $\mho$. It will also be useful later in Corollary 11.7.13. We suggested in Remark 11.3.7 that $\multimap$ is a kind of pattern-matching; by that view, what we do now is pattern-matching on a universally quantified atom:

Proposition 11.3.9. $\forall a.(a{\uparrow_\Pi}\multimap s{\uparrow_\Pi}) = (\lambda a.s){\uparrow_\Pi}$.

Proof. We prove two subset inclusions:

— *Proof that* $\forall a.(a{\uparrow_\Pi}\multimap s{\uparrow_\Pi}) \subseteq (\lambda a.s){\uparrow_\Pi}$. By condition 4 of Definition 10.3.7 $(\lambda a.s)a{\to_\Pi}s$, so by Lemma 11.7.4 $s{\uparrow_\Pi} \subseteq ((\lambda a.s)a){\uparrow_\Pi} \overset{\text{Lem 11.3.8}}{=} (\lambda a.s){\uparrow_\Pi}\bullet a{\uparrow_\Pi}$. It follows by Lemma 11.3.4 that $a{\uparrow_\Pi}\multimap s{\uparrow_\Pi} \subseteq (\lambda a.s){\uparrow_\Pi}$. By condition 1 of Definition 10.3.1 $a\#\lambda a.s$ so by Theorem 2.3.1 $a\#(\lambda a.s){\uparrow_\Pi}$, and therefore by Lemma 11.3.5 $\forall a.(a{\uparrow_\Pi}\multimap s{\uparrow_\Pi}) \subseteq (\lambda a.s){\uparrow_\Pi}$.
— *Proof that* $(\lambda a.s){\uparrow_\Pi} \subseteq \forall a.(a{\uparrow_\Pi}\multimap s{\uparrow_\Pi})$. By condition 4 of Definition 10.3.7 $(\lambda a.s)a{\to_\Pi}s$, so by Lemma 11.3.6 $\lambda a.s \in a{\uparrow_\Pi}\multimap s{\uparrow_\Pi}$. By Lemma 11.3.5 $\lambda a.s \in \forall a.(a{\uparrow_\Pi}\multimap s{\uparrow_\Pi})$, and by condition 1 of Definition 11.1.3 $(\lambda a.s){\uparrow_\Pi} \subseteq \forall a.(a{\uparrow_\Pi}\multimap s{\uparrow_\Pi})$.

$\square$

11.4. The left adjoint $p[a{\mapsto}u]$ to the $\mho$-action $p[u{\hookleftarrow}a]$

By Proposition 11.1.8 a finitely-supported intersection of points is a point. This suggests that we could build a left adjoint to $[u{\hookleftarrow}a]$ on points by taking a suitable intersection. We do this in Definition 11.4.1.
This left adjoint turns out to be very well-behaved. It has interesting characterisations (Subsection 11.4.2) which give us strong proof-methods for reasoning on it. Furthermore it is a σ-action; this is Proposition 11.5.6. So $points_\Pi$ is both an $\mho$-algebra and a σ-algebra.[39]
Even better, the σ-action commutes with $\text{-}^\bullet$ from Definition 11.7.1, which is key to how points are used to generate compact sets; this is part 1 of Theorem 11.7.8. Thus we can study the behaviour of

[39] This fact is most likely a special case of a general result which deserves a paper written in its own right. For this paper, we are simply grateful for the fact and we press on to make good use of it.

substitution on open sets by understanding the behaviour of the left adjoint to the $\mho$-algebra action of points.

In short, most of the rest of this section depends on Definition 11.4.1 and the results that follow it in this subsection.

11.4.1. Basic definition. Recall from Definition 11.2.3 the $\mho$-action on points $p[u\leftarrow\!\!\dashv a]$. We can build a left adjoint for it:

DEFINITION 11.4.1. Given $p \subseteq |\mho|$ with finite support and $u \in |\mho|$, define $p[a\mapsto u]$ by:

$$p[a\mapsto u] = \bigcap\{q \in |points_{\Pi}| \mid \text{И}c.((c\ a)\cdot p \subseteq q[u\leftarrow\!\!\dashv c])\}$$

By Proposition 11.1.8, Definition 11.4.1 does indeed define a point.

REMARK 11.4.2. Definition 11.4.1 looks like a repeat of Definition 3.4.1, but they are not quite the same because p and q above have the same type (subsets of $|\mho|$) whereas in Definition 3.4.1 p and X have different types (a point and a set of points, respectively).

Points here have finite support, so the proof of Proposition 11.4.3 is (almost) a replay of the proof of part 2 of Proposition 3.4.2 (only for subset inclusion instead of sets membership):

PROPOSITION 11.4.3. *If $a\#u, q$ then $p \subseteq q[u\leftarrow\!\!\dashv a]$ if and only if $p[a\mapsto u] \subseteq q$.*

Proof. Suppose $a\#u, q$. From Definition 11.4.1, $p[a\mapsto u] \subseteq q$ if and only if $\text{И}c.(c\ a)\cdot p \subseteq q[u\leftarrow\!\!\dashv c]$. By Corollary 2.1.10 $(c\ a)\cdot u = u$ and $(c\ a)\cdot q = q$, so (applying $(c\ a)$ to both sides of the subset inclusion) this is if and only if $\text{И}c.p \subseteq q[u\leftarrow\!\!\dashv a]$, that is: $p \subseteq q[u\leftarrow\!\!\dashv a]$. □

LEMMA 11.4.4. *Suppose $p \subseteq |\mho|$ is finitely supported and $b\#p$. Then $p[a\mapsto u] = ((b\ a)\cdot p)[b\mapsto u]$.*

As a corollary, if $a\#u$ then $a\#p[a\mapsto u]$.

Proof. Suppose c is fresh (so $c\#p$). By Theorem 2.3.9 it suffices to show that $(c\ a)\cdot p \subseteq q[u\leftarrow\!\!\dashv c]$ if and only if $(c\ b)\cdot((b\ a)\cdot p) \subseteq q[u\leftarrow\!\!\dashv c]$. It would suffice to prove that $(c\ a)\cdot p = (c\ b)\cdot((b\ a)\cdot p)$. This follows from Corollary 2.1.10 and our assumption that $b\#p$.

The corollary follows using Corollary 2.1.10. □

11.4.2. Two characterisations of $p[a\mapsto u]$

DEFINITION 11.4.5. If $p \subseteq |\mho|$ and $u \in |\mho|$ define

$$p[a{:=}u] = \bigcup\{s[a{:=}u]{\uparrow_{\Pi}} \mid s \in p\}.$$

By Proposition 11.1.8 and Lemma 11.2.2, $p[a{:=}u]$ is a point.

For the rest of this subsection, we assume $p, q \in |points_{\Pi}|$.

LEMMA 11.4.6. *If $a\#u, q$ then $p[a\mapsto u] \subseteq q$ if and only if $p[a{:=}u] \subseteq q$.*

As a corollary, if $a\#u$ then $p[a{:=}u] \subseteq p[a\mapsto u]$.

Proof. By Proposition 11.4.3 $p[a\mapsto u] \in q$ if and only if $p \subseteq q[u\leftarrow\!\!\dashv a]$. From Proposition 3.3.2 and condition 1 of Definition 11.1.3 this happens if and only if $p[a{:=}u] \subseteq q$.

The corollary follows since $p[a\mapsto u] \subseteq p[a\mapsto u]$ and by Lemma 11.4.4 (the corollary part) $a\#p[a\mapsto u]$. □

COROLLARY 11.4.7. *If $a\#u$ then*

$$p[a\mapsto u] = \bigcap\{q \mid a\#q \wedge p[a{:=}u]{\subseteq}q\}. \qquad \textit{(Characterisation 1)}$$

Proof. If $a\#q$ and $p[a{:=}u] \subseteq q$ then by Lemma 11.4.6 also $p[a\mapsto u] \subseteq q$. Therefore

$$p[a\mapsto u] \subseteq \bigcap\{q \mid a\#q \wedge p[a{:=}u] \subseteq q\}.$$

Furthermore by Lemma 11.4.4 $a\#p[a\mapsto u]$ and by Lemma 11.4.6 $p[a{:=}u] \subseteq p[a\mapsto u]$. Therefore

$$\bigcap\{q \mid a\#q \wedge p[a{:=}u] \subseteq q\} \subseteq p[a\mapsto u].$$

$\square$

Recall from Definition 2.5.1 the notion of $\text{И}a.p$ (the И-quantifier, for sets).

LEMMA 11.4.8. *If $p \in |points_\Pi|$ then also $\text{И}a.p \in |points_\Pi|$.*

Proof. We verify the conditions of Definition 11.1.3:

(1) Suppose $s \in \text{И}a.p$ and $s{\to_\Pi}t$. Then $\text{И}b.(b\ a)\cdot s \in p$. By equivariance of Π (Definition 10.3.7) $(b\ a)\cdot s{\to_\Pi}(b\ a)\cdot t$ (for any b). It follows by condition 1 of Definition 11.1.3 that $\text{И}b.(b\ a)\cdot t \in p$, so that $b \in \text{И}a.p$.
(2) Consider some fresh c (so $c\#p$) and consider $s \in \text{И}a.p$ and some fresh b (so $b\#p, s$), so that $(b\ a)\cdot s \in p$. By condition 2 of Definition 11.1.3, $c \sharp_\sigma p$, so that $\forall u.((b\ a)\cdot s)[c{:=}u] \in p$. It follows that $\forall u.(b\ a)\cdot(s[c{:=}u]) \in p$, and therefore $\forall u.s[c{:=}u] \in \text{И}a.p$.

$\square$

Lemma 11.4.9 gives a striking characterisation connecting the left adjoint to the $\mho$-action, the pointwise substitution action, and the И-quantifier for sets (Definitions 11.4.1, 11.4.5, and 2.5.1):

LEMMA 11.4.9. *If $a\#u$ then $p[a{:=}u] \subseteq \text{И}a.(p[a{:=}u])$. As a corollary, if $a\#u$ then*

$$p[a\mapsto u] = \text{И}a.(p[a{:=}u]). \qquad \textit{(Characterisation 2)}$$

Proof. Suppose $s \in p[a{:=}u]$. This means there is $s'{\in}p$ such that $s'[a{:=}u]{\to_\Pi}s$. By Lemma 3.2.5 $a\#s'[a{:=}u]$ and it follows for any fresh c that $s'[a{:=}u]{\to_\Pi}(c\ a)\cdot s$, so that $(c\ a)\cdot s \in p[a{:=}u]$. Thus, $s \in \text{И}a.(p[a{:=}u])$.

The corollary follows from Lemma 2.5.3 and Corollary 11.4.7. $\square$

11.4.3. Additional lemmas about the σ-action as an adjoint. Lemmas 11.4.10 and 11.4.11 describe a unit and counit style interaction between $[a\mapsto u]$ and $[u\leftarrow a]$ acting on points. We will not use these lemmas later—we will use the adjunction result they come from, Proposition 11.4.3, directly instead.

The lemmas are still worth looking at, because they are subject to freshness side-conditions and so are not quite exactly what one might assume.

LEMMA 11.4.10. *If $a\#u$ then $p \subseteq p[a\mapsto u][u\leftarrow a]$.*

Proof. By Lemma 11.4.4 (the corollary part) $a\#p[a\mapsto u]$. It is a fact that $p[a\mapsto u] \subseteq p[a\mapsto u]$, therefore by Proposition 11.4.3 $p \subseteq p[a\mapsto u][u\leftarrow a]$. $\square$

We cannot prove $p[u\leftarrow a][a\mapsto u] \subseteq p$ from $p[u\leftarrow a] \subseteq p[u\leftarrow a]$ using Proposition 11.4.3, because we do not necessarily know $a\#p[u\leftarrow a]$. Interestingly, a reverse inclusion does still hold:

LEMMA 11.4.11. *If $a\#p$ then $p \subseteq p[u\leftarrow a][a\mapsto u]$.*

Proof. Suppose $s \in p$. Using Lemma 11.4.4 to rename if necessary, we may assume $a\#s, u$. So $s[a{:=}u] = s$, so that $s \in p[u\leftarrow a]$ and so that $s \in p[u\leftarrow a][a{:=}u] \overset{\text{L 11.4.6}}{\subseteq} p[u\leftarrow a][a\mapsto u]$. $\square$

11.5. The left-adjoint $p[a\mapsto u]$ as a σ-action on points

11.5.1. It is indeed a σ-action. We prove Proposition 11.5.6, that $p[a\mapsto u]$ is indeed a σ-action on points.

LEMMA 11.5.1. *Suppose $p \in |points_\Pi|$ and $a\#p$. Then $p[a\mapsto u] = p$.*

Proof. Using Lemma 11.4.4 and Corollary 2.1.10 assume without loss of generality that $a\#u$ as well as $a\#p$.

By Lemma 11.2.6 $p \subseteq p[u\leftarrowtail a]$ so by Proposition 11.4.3 $p[a\mapsto u] \subseteq p$.

Now suppose $s \notin p[a\mapsto u]$. We will show that $s \notin p$.

Unpacking Definition 11.4.1, $s \notin p[a\mapsto u]$ implies that there exists $q \in |points_\Pi|$ such that $s \notin q$ and for fresh c (so $c\#p, q, u, s$) $(c\ a)\cdot p \subseteq q[u\leftarrowtail c]$. Now by $(\sigma\#)$ $s[c{:=}u] = s$. Thus if $s \notin q$ then by Proposition 3.4.2 also $s \notin q[u\leftarrowtail c]$. Therefore $s \notin (c\ a)\cdot p$. By Corollary 2.1.10 since $a\#p$ and $c\#p$ also $(c\ a)\cdot p = p$, so $s \notin p$ as required. $\square$

Lemma 11.5.2. *If $b\#p$ then $p[a\mapsto b] = (b\ a)\cdot p$.*

Proof. By Lemma 11.4.9 $s \in p[a\mapsto b]$ if and only if $\text{И}c.(c\ a)\cdot s \in p[a{:=}b]$, and by Definitions 11.2.1 and 11.4.5 this is if and only if $\text{И}c.\exists s'{\in}p.s'[a{:=}b]{\to_\Pi}(c\ a)\cdot s$.

Now by assumption $b\#p$ so if $b \in supp(s')$ then by condition 2 of Definition 11.1.3 also $s'[b{:=}a] \in p$. So we may assume without loss of generality of the '$\exists s'{\in}p$' above that the s' chosen satisfies $b\#s'$, so that $s'[a{:=}b] = (b\ a)\cdot s'$.

Thus this is if and only if $\text{И}c.\exists s'{\in}p.(b\ a)\cdot s'{\to_\Pi}(c\ a)\cdot s$. Rearranging the permutations, this is if and only if $\text{И}c.\exists s'{\in}p.(b\ a)\cdot((c\ b)\cdot s'){\to_\Pi}s$.

Again, since $c, b\#p$, by Corollary 2.1.10 $(c\ b)\cdot p = p$ so that $s'{\in}p$ if and only if $(c\ b)\cdot s'{\in}p$.

Thus this is if and only if $\text{И}c.\exists s'{\in}p.(b\ a)\cdot s'{\to_\Pi}s$, and by condition 1 of Definition 11.1.3 this is if and only if $s \in (b\ a)\cdot p$. $\square$

Remark 11.5.3. Lemma 11.5.2 is remarkable. There is no reason to expect that $p[a\mapsto b] = (b\ a)\cdot p$ should hold—for contrast, we needed to impose this as condition 2 when we constructed $pow_\sigma(\mathcal{P})$ in Definition 3.4.5. Here, it works without requiring conditions.

Corollary 11.5.4 is a repeat of Corollary 3.4.7, but for points. We will use it in Proposition 11.5.6:

Corollary 11.5.4. *$p[a\mapsto a] = p$.*

Proof. Choose any $b\#p$. By Lemma 11.4.4 $p[a\mapsto a] = ((b\ a)\cdot p)[b\mapsto a]$. By Proposition 2.3.3 $a\#(b\ a)\cdot X$, and by Lemma 11.5.2 $((b\ a)\cdot p)[b\mapsto a] = (b\ a)\cdot((b\ a)\cdot p) = X$. $\square$

Lemma 11.5.5. *If $a\#v$ then $p[a\mapsto u][b\mapsto v] = p[b\mapsto v][a\mapsto u[a\mapsto v]]$.*

Proof. Using Lemma 11.4.4 assume without loss of generality that $a\#u$ and $b\#v$. Then the result follows using Proposition 11.4.3 and Corollary 11.2.5, from $(\mathfrak{v}\sigma)$. $\square$

Proposition 11.5.6 does not hold in the general case of $F(\mathcal{D})$ from Definition 7.2.1, but it holds specifically for $points_\Pi$. The underlying reason this happens is Proposition 11.1.8, which allows us to build (finitely-supported) intersections of points and so construct $p[a\mapsto u]$ in Definition 11.4.1:

Proposition 11.5.6. *$points_\Pi$ with the action $[a\mapsto u]$ from Definition 11.4.1 is indeed a σ-algebra.*

Proof. The interesting part is to check the axioms of Figure 1:

— $(\sigma\mathbf{id})$ is Corollary 11.5.4.
— $(\sigma\#)$ is Lemma 11.5.1.
— $(\sigma\alpha)$ is Lemma 11.4.4.
— $(\sigma\sigma)$ is Lemma 11.5.5.

$\square$

11.5.2. The σ-action distributes over union and subset

Lemma 11.5.7. *Suppose $\mathcal{P} \subseteq |points_\Pi|$ is strictly finitely supported (Subsection 2.4.2). Then*

$$(\bigcup \mathcal{P})[a\mapsto u] = \bigcup\{p[a\mapsto u] \mid p \in \mathcal{P}\}.$$

Proof. We use Lemma 11.4.4 to assume without loss of generality that $a\#u$. Take any $r \in |points_\Pi|$ such that $a\#r$. We reason as follows:

$$\begin{array}{rll} (\bigcup\mathcal{P})[a\mapsto u] \subseteq r & \Leftrightarrow \bigcup\mathcal{P} \subseteq r[u\leftarrow a] & \text{Proposition 11.4.3} \\ & \Leftrightarrow \forall p{\in}\mathcal{P}.p \subseteq r[u\leftarrow a] & \text{Fact of sets} \\ & \Leftrightarrow \forall p{\in}\mathcal{P}.p[a\mapsto u] \subseteq r & \text{Proposition 11.4.3} \\ & \Leftrightarrow \bigcup\{p[a\mapsto u] \mid p \in \mathcal{P}\} \subseteq r & \text{Fact of sets} \end{array}$$

By Lemma 11.4.4 (the corollary part) $a\#(\bigcup\mathcal{P})[a\mapsto u]$ and $a\#p[a\mapsto u]$ for every $p \in \mathcal{P}$ so that by Lemma 2.4.3 also $a\#\bigcup\{p[a\mapsto u] \mid p \in \mathcal{P}\}$. Taking $r = (\bigcup\mathcal{P})[a\mapsto u]$ and $r = \bigcup\{p[a\mapsto u] \mid p \in \mathcal{P}\}$ we obtain two subset inclusions and thus an equality. □

Remark 11.5.8. Lemma 11.5.7 does not give us distributivity of of substitution over sets intersection of points. This is why, as noted in Remark 11.3.2, we do not consider an operation $p \vee q$; it would not satisfy $\cancel{(p\vee q)[a\mapsto u] = p[a\mapsto u]\vee q[a\mapsto u]}$. To get this kind of property we need the topologies, developed below. See in particular Corollary 11.7.9.

Lemma 11.5.9. *If $p \subseteq q$ then $p[a\mapsto u] \subseteq q[a\mapsto u]$.*

Proof. Using Lemma 11.5.7, since $p \subseteq q$ if and only if $p \cup q = q$. □

11.5.3. Relating $\forall a.p$ and the σ-action $p[a\mapsto u]$. Recall that for $p \in points_\Pi$ we defined $\forall a.p$ in Definition 11.3.1 and $p[a\mapsto u]$ in Definition 11.4.1. (We proved that $[a\mapsto u]$ is a σ-action, as the notation suggests, in Proposition 11.5.6.)

Definition 11.3.1 defined $\forall a.p = \bigcap\{r \in |points_\Pi| \mid p \subseteq r \wedge a\#r\}$. There is nothing in this to *a priori* suggest that this should be a universal quantification. But it is: we prove it in Proposition 11.5.12.

Lemma 11.5.10. *$a\#\forall a.p$.*

As a corollary, $supp(\forall a_1, \dots, a_n.p) \subseteq supp(p)\setminus\{a_1, \dots, a_n\}$.

Proof. The first part is from Definition 11.3.1 and Lemma 2.4.3. The corollary follows using the first part and Theorem 2.3.1. □

Corollary 11.5.11. *$a \sharp_\sigma \forall a.p$ and as a corollary, if $s \in \forall a.p$ then $\forall u.(s[a{:=}u] \in \forall a.p)$.*

Proof. By Lemma 11.5.10 $a\#p$, by Lemma 11.3.3 $\forall a.p$ is a point, and by Proposition 11.1.6 $a \sharp_\sigma \forall a.p$. The corollary follows from Definition 11.1.1. □

We can now prove that $\forall a.p$ is indeed a universal quantification over $u{\in}|\mho|$. If the reader is puzzled by the use of $\bigcup$ here—should it not be $\bigcap$?—remember that with points things are dual and so 'upside-down' (see Lemma 11.7.6 and part 2 of Proposition 11.7.11).

Proposition 11.5.12. $\forall a.p = \bigcup_{u\in|\mho|} p[a\mapsto u]$.

Proof.— *The left-to-right inclusion.* By construction $p \subseteq \bigcup_{u\in|\mho|} p[a\mapsto u]$ (we take $u = a$ and use Corollary 11.5.4). In addition, using Lemma 11.4.4 and Theorem 2.3.1 it can be proved that $a\#\bigcup_{u\in|\mho|} p[a\mapsto u]$. We use part 4 of Lemma 11.3.5.

— *The right-to-left inclusion.* By construction in Definition 11.3.1 $p \subseteq \forall a.p$, so by Lemmas 11.5.9 and 11.5.1 $p[a\mapsto u] \subseteq \forall a.p$ for every u.

□

11.6. How the σ-action on points commutes

The set of points $points_\Pi$ has plenty of structure. It is a nominal set, it has an $\mho$-action $p[u\leftarrow a]$ (Corollary 11.2.5), a σ-action $p[a\mapsto u]$ (Proposition 11.5.6) and a subsidiary pointwise version $p[a{:=}u]$ (Definition 11.4.5). It is a fresh semi-lattice (a top element, $\wedge$, and $\forall$; see Remark 11.1.4 and Definition 11.3.1) and has $\bullet$ and $-\!\!\bullet$ (Definition 11.3.1) and even $\text{И}a.p$ a sets version of the И-quantifer (Lemma 11.4.8). There is also a map from $|\mho|$ to points given by $s \in |\mho|$ maps to $s{\uparrow}_\Pi$ (Definition 11.2.1).

In this subsection we consider useful ways in which the σ-action commutes with some of this structure. These commutation results will later be useful in proving that sets of points have the structure of an impredicative lattice with $\forall$ and $\bullet$.

LEMMA 11.6.1. $s{\uparrow_\Pi}[a{\mapsto}u] = s[a{:=}u]{\uparrow_\Pi}$.

Proof. Using $(\sigma\alpha)$ and Lemma 11.4.4 assume without loss of generality that $a\#u$. We prove two subset inclusions.

— *Proof that* $s{\uparrow_\Pi}[a{\mapsto}u] \subseteq s[a{:=}u]{\uparrow_\Pi}$. By Lemma 3.2.5 $a\#s[a{:=}u]$ and by Theorem 2.3.1 also $a\#s[a{:=}u]{\uparrow_\Pi}$. Thus by Lemma 11.4.6 to it suffices to prove $s{\uparrow_\Pi}[a{:=}u] \subseteq s[a{:=}u]{\uparrow_\Pi}$. Suppose $s{\to_\Pi}t$. By condition 3 of Definition 10.3.7 $s[a{:=}u]{\to_\Pi}t[a{:=}u]$. The result follows.
— *Proof that* $s[a{:=}u]{\uparrow_\Pi} \subseteq s{\uparrow_\Pi}[a{\mapsto}u]$.
By condition 1 of Definition 11.1.3 it suffices to note that $s[a{:=}u] \in s{\uparrow_\Pi}[a{\mapsto}u]$.

□

LEMMA 11.6.2.— $(p\bullet q)[a{:=}u] = p[a{:=}u]\bullet q[a{:=}u]$.
— $(\text{И}a.p)\bullet(\text{И}a.q) = \text{И}a.(p\bullet q)$ *(Definition 2.5.1; Lemma 11.4.8).*

Proof. From Definitions 11.3.1 and 11.4.5, we have that $r \in (p\bullet q)[a{:=}u]$ when there exist $s \in p$ and $t \in q$ such that $(st)[a{:=}u]{\to_\Pi}r$, and that $r \in p[a{:=}u]\bullet q[a{:=}u]$ when there exist $s \in p$ and $t \in q$ such that $s[a{:=}u]\,t[a{:=}u]{\to_\Pi}r$. By $(\sigma\bullet)$ $(st)[a{:=}u] = s[a{:=}u]\,t[a{:=}u]$.

For the second part, $r \in (\text{И}a.p)(\text{И}a.q)$ when there exist s and t such that $\text{И}b.(b\ a)\cdot s \in p$ and $\text{И}b.(b\ a)\cdot t \in q$ and $r = st$. It is a fact that the И-quantifier distributes over conjunction [Gab11a, Theorem 6.6], so also $\text{И}b.(b\ a)\cdot(st) \in p\bullet q$. The result follows. □

COROLLARY 11.6.3. *Suppose* $p, q \in |points_\Pi|$. *Then:*

— $(p\wedge q)[a{\mapsto}u] = p[a{\mapsto}u]\wedge q[a{\mapsto}u]$.
— $(p\bullet q)[a{\mapsto}u] = p[a{\mapsto}u]\bullet q[a{\mapsto}u]$.

Proof. For the first part, by Definition 11.3.1 $p\wedge q = p\cup q$. We use Lemma 11.5.7.

For the second part, we use Lemma 11.4.4 to assume without loss of generality that $a\#u$. By Lemma 11.4.9 $(p\bullet q)[a{\mapsto}u] = \text{И}a.((p\bullet q)[a{:=}u])$ and $p[a{\mapsto}u]\bullet q[a{\mapsto}u] = (\text{И}a.(p[a{:=}u]))\bullet(\text{И}a.(q[a{:=}u]))$. We use Lemma 11.6.2. □

Proposition 11.6.4 is the key technical result to proving $(\sigma{-}\!\bullet)$ valid in Corollary 11.8.18:

PROPOSITION 11.6.4. *Suppose* $u \in |\mho|$ *and* $b\#u$. *Suppose* $p \in |points_\Pi|$. *Then* $(b{\uparrow_\Pi}{-\!\bullet}\,p)[a{\mapsto}u] = b{\uparrow_\Pi}{-\!\bullet}(p[a{\mapsto}u])$.

Proof. For the right-to-left inclusion $b{\uparrow_\Pi}{-\!\bullet}(p[a{\mapsto}u]) \subseteq (b{\uparrow_\Pi}{-\!\bullet}\,p)[a{\mapsto}u]$ we reason as follows:

$$\begin{array}{rll}
b{\uparrow_\Pi}{-\!\bullet}(p[a{\mapsto}u]) \subseteq (b{\uparrow_\Pi}{-\!\bullet}\,p)[a{\mapsto}u] & \Leftrightarrow p[a{\mapsto}u] \subseteq (b{\uparrow_\Pi}{-\!\bullet}\,p)[a{\mapsto}u]\bullet(b{\uparrow_\Pi}) & \text{Lemma 11.3.4} \\
& \Leftrightarrow p[a{\mapsto}u] \subseteq (b{\uparrow_\Pi}{-\!\bullet}\,p)[a{\mapsto}u]\bullet(b{\uparrow_\Pi}[a{\mapsto}u]) & \text{Lemma 11.5.1} \\
& \Leftrightarrow p[a{\mapsto}u] \subseteq ((b{\uparrow_\Pi}{-\!\bullet}\,p)\bullet b{\uparrow_\Pi})[a{\mapsto}u] & \text{Corollary 11.6.3} \\
& \Leftarrow p[a{\mapsto}u] \subseteq p[a{\mapsto}u] & \text{Ls 11.5.9 \& 11.3.4}
\end{array}$$

For the left-to-right inclusion $(b{\uparrow_\Pi}{-\!\bullet}\,p)[a{\mapsto}u] \subseteq b{\uparrow_\Pi}{-\!\bullet}(p[a{\mapsto}u])$, using Lemma 11.4.4 to rename if necessary assume $a\#u$. Using Theorem 2.3.1 also note that $a\#b{\uparrow_\Pi}{-\!\bullet}(p[a{\mapsto}u])$. Thus by Lemma 11.4.6 and part 3 of Lemma 11.3.4 it suffices to prove that $(b{\uparrow_\Pi}{-\!\bullet}\,p)[a{:=}u] \subseteq b{\uparrow_\Pi}{-\!\bullet}(p[a{:=}u])$.

Unpacking Definitions 11.3.1 (for ${-\!\bullet}$) 11.2.1 (for ${\uparrow_\Pi}$) and 11.4.5 (for $[a{:=}u]$ on points), this simplifies to showing that for all s,

$$\exists s'.(s'b{\in}p \wedge s{=}s'[a{:=}u]) \Rightarrow \exists t'.(t'{\in}p \wedge sb{=}t'[a{:=}u]).$$

So suppose s' is such that $s'b{\in}p$ and $s{=}s'[a{:=}u]$. Take $t' = s'b$. So $t'[a{:=}u] = (s'b)[a{:=}u] = sb$ as required. □

11.7. Operations on sets of points

11.7.1. Basic definitions

DEFINITION 11.7.1. Suppose $p, q \in |points_\Pi|$ and $s \in |\mho|$. Define the following operations:

$$\boxed{\begin{aligned} p^\bullet &= \{q \in |points_\Pi| \mid p \subseteq q\} \\ p \circ q &= (p \bullet q)^\bullet \end{aligned}}$$

REMARK 11.7.2. $p^\bullet$ and $\circ$ generate sets of points. We use these to build a topology and a nominal spectral space with $\circ$ over $points_\Pi$ and so to prove Theorem 11.9.5 (completeness).

$p \circ q$ is a *combination operator* in the sense of Definition 9.2.1, so that we get operations $\bullet$ and $-\!\!\bullet$ on sets of points from Definition 9.2.3.

Thus we have two operations named $\bullet$; one on points from Definition 11.3.1 and one on sets of points from Definition 11.7.1. Similarly, we have two operations named $-\!\!\bullet$. It will always be clear from context which is meant, and they are related by Proposition 11.7.11.

REMARK 11.7.3. It might be worth mentioning that $p^\bullet$ from Definition 11.7.1 is not a repeat of $x^\bullet$ from Definition 6.3.1:

— In Definition 6.3.1 we assumed an underlying nominal distributive lattice with $\forall$.
— Definition 11.7.1 is constructed using $points_\Pi$. Now we consider Definition 11.3.1 and see that if we ignore $\bullet$ and $-\!\!\bullet$ then $points_\Pi$ is almost a nominal distributive lattice with $\forall$—but it lacks a disjunction. We could reasonably call it a semilattice with $\forall$.

So $points_\Pi$ is a specific structure with its own properties.

The proof of Lemma 11.7.4 is fairly simple, by unfolding definitions. However it is very important; for instance it is the final step in the proof of Completeness in Theorem 11.9.5.

LEMMA 11.7.4. *The following conditions are equivalent:*

$$s{\uparrow}_\Pi^\bullet \subseteq t{\uparrow}_\Pi^\bullet \quad\Leftrightarrow\quad s{\uparrow}_\Pi \in t^\bullet \quad\Leftrightarrow\quad t \in s{\uparrow}_\Pi \quad\Leftrightarrow\quad s{\to}_\Pi t \quad\Leftrightarrow\quad t{\uparrow}_\Pi \subseteq s{\uparrow}_\Pi$$

Proof. Suppose $s{\uparrow}_\Pi^\bullet \subseteq t{\uparrow}_\Pi^\bullet$. Then in particular $s{\uparrow}_\Pi \in t{\uparrow}_\Pi^\bullet$. This means $t \in s{\uparrow}_\Pi$, and so $s{\to}_\Pi t$. It follows by condition 1 of Definition 11.1.3 that $t{\uparrow}_\Pi \subseteq s{\uparrow}_\Pi$.

Now suppose $t{\uparrow}_\Pi \subseteq s{\uparrow}_\Pi$ and $p \in s{\uparrow}_\Pi^\bullet$. Then $s \in p$, and by condition 1 of Definition 11.1.3 $s{\uparrow}_\Pi \subseteq p$, so that $t \in p$ and $p \in t{\uparrow}_\Pi^\bullet$. □

COROLLARY 11.7.5. $s{\uparrow}_\Pi^\bullet = t{\uparrow}_\Pi^\bullet$ *if and only if* $s =_\Pi t$.

Proof. From Lemma 11.7.4. □

Lemma 11.7.6 is in the spirit of Lemma 6.3.2:

LEMMA 11.7.6. *Suppose* $p, q \in |points_\Pi|$. *Then the following conditions are equivalent:*

$$p \in q^\bullet \quad\Leftrightarrow\quad q \subseteq p \quad\Leftrightarrow\quad p^\bullet \subseteq q^\bullet$$

Furthermore, if $s \in |\mho|$ *then* $p \in s{\uparrow}_\Pi^\bullet \Leftrightarrow s \in p$.

Proof. Unpacking Definition 11.7.1, $p \in q^\bullet$ does imply $q \subseteq p$. It is a fact of sets that if $q \subseteq p$ then $p \subseteq p'$ implies $q \subseteq p'$. Finally, if $p^\bullet \subseteq q^\bullet$ then since $p \in p^\bullet$, also $p \in q^\bullet$.

The corollary follows from Lemma 11.7.4. □

COROLLARY 11.7.7. *The assignment* $p \longmapsto p^\bullet$ *is injective from* $|points_\Pi|$ *to* $pow(points_\Pi)$. *As a corollary,* $supp(p^\bullet) = supp(p)$.

Proof. The first part follows using Lemma 11.7.6. The corollary follows by part 3 of Theorem 2.3.1. □

11.7.2. Commutation properties. Recall $q[a\mapsto u]$ from Definition 11.4.1, $p[u\leftarrow a]$ from Definition 3.3.1, and (since by Corollary 11.2.5 $points_\Pi$ is an $\mho$-algebra) $p^\bullet[a\mapsto u]$ from Definition 3.4.1.

Theorem 11.7.8 is fairly easy to prove, but part 1 of it is key. It relates the natural σ-action $p^\bullet[a\mapsto u]$ to the σ-action on points $p[a\mapsto u]$ from Proposition 11.5.6. Compare Theorem 11.7.8 with Lemma 6.4.1:

THEOREM 11.7.8. *Suppose $q \in |points_\Pi|$ and $u \in |\mho|$.*

(1) $q^\bullet[a\mapsto u] = (q[a\mapsto u])^\bullet$.
As a corollary taking $p = s\!\uparrow_\Pi$, $s\!\uparrow^\bullet_\Pi[a\mapsto t] = s[a{:=}t]\!\uparrow^\bullet_\Pi$.
(2) $\pi\cdot(q^\bullet) = (\pi\cdot q)^\bullet$.

Proof. Consider some p; we wish to show that $p \in q^\bullet[a\mapsto u] \Leftrightarrow p \in (q[a\mapsto u])^\bullet$. By Lemmas 3.4.3 and 11.4.4 we may α-rename a in $q^\bullet[a\mapsto u]$ and $q[a\mapsto u]$ to assume without loss of generality that $a\#u, p$. We reason as follows; we use part 2 of Proposition 3.4.2 because by Corollary 11.1.7 p and q have finite support:

$$\begin{aligned} p \in q^\bullet[a\mapsto u] &\Leftrightarrow p[u\leftarrow a] \in q^\bullet && \text{Proposition 3.4.2}\\ &\Leftrightarrow q \subseteq p[u\leftarrow a] && \text{Definition 11.7.1}\\ &\Leftrightarrow q[a\mapsto u] \subseteq p && \text{Proposition 11.4.3}\\ &\Leftrightarrow p \in (q[a\mapsto u])^\bullet && \text{Definition 11.7.1}\end{aligned}$$

The corollary follows using Lemma 11.6.1.

The second part is proved by similar calculations, or directly from Theorem 2.3.1. ☐

Corollary 11.7.9 describes how the σ-action interacts with sets union (see also Proposition 11.8.4):

COROLLARY 11.7.9. *Suppose $\mathcal{P} \subseteq |points_\Pi|$ is strictly finitely supported. Then*

$$(\bigcup_{p\in\mathcal{P}} p^\bullet)[a\mapsto u] = \bigcup_{p\in\mathcal{P}} (p[a\mapsto u])^\bullet.$$

Proof. From Lemma 5.1.1 and Theorem 11.7.8. ☐

COROLLARY 11.7.10.*(1)* *$a\#p^\bullet$ then $(p^\bullet)[a\mapsto u] = p^\bullet$.*
(2) *If $b\#p^\bullet$ then $((b\ a)\cdot p^\bullet)[b\mapsto u] = p^\bullet[a\mapsto u]$.*

Proof. For the first part, suppose $a\#p^\bullet$. By Corollary 11.7.7 $a\#p$. We use Theorem 11.7.8 and Lemma 11.5.1.

For the second part we reason similarly using Theorem 11.7.8 and Lemma 11.4.4. ☐

PROPOSITION 11.7.11.*(1)* $p^\bullet \cap q^\bullet = (p\wedge q)^\bullet$.
(2) $\bigcap^{\#a} p^\bullet = (\forall a.p)^\bullet$.
(3) $p^\bullet \bullet q^\bullet = (p\bullet q)^\bullet$.
(4) $q^\bullet \multimap\!\!\bullet\, p^\bullet = (q\multimap\!\!\bullet\, p)^\bullet$.

Proof. We consider each case in turn; with what we have proved so far, the calculations are routine. u will range over elements of $|\mho|$:

(1) We reason as follows:

$$\begin{aligned} r \in p^\bullet \cap q^\bullet &\Leftrightarrow p \subseteq r \text{ and } q \subseteq r && \text{Definition 11.7.1}\\ &\Leftrightarrow p \cup q \subseteq r && \text{Fact}\\ &\Leftrightarrow p \in (p\wedge q)^\bullet && \text{Definition 11.7.1}\end{aligned}$$

(2) We reason as follows:

$$\begin{aligned} \textstyle\bigcap^{\#a} p^\bullet &= \textstyle\bigcap_{u\in|\mho|} p^\bullet[a\mapsto u] && \text{Definition 5.2.1}\\ &= \textstyle\bigcap_{u\in|\mho|} (p[a\mapsto u])^\bullet && \text{Theorem 11.7.8}\\ &= (\textstyle\bigcup_{u\in|\mho|} p[a\mapsto u])^\bullet && \text{Fact of Def 11.7.1}\\ &= (\forall a.p)^\bullet && \text{Proposition 11.5.12}\end{aligned}$$

(3) We reason as follows:

$$
\begin{array}{rll}
r \in p^{\bullet}{\bullet}q^{\bullet} & \Leftrightarrow \exists p'{\in}p^{\bullet}, q'{\in}q^{\bullet}. r \in p'{\circ}q' & \text{Proposition 9.2.6} \\
& \Leftrightarrow \exists p', q'. p \subseteq p' \wedge q \subseteq q' \wedge r \in p'{\circ}q' & \text{Definition 11.7.1} \\
& \Leftrightarrow \exists p', q'. p \subseteq p' \wedge q \subseteq q' \wedge p'{\bullet}q' \subseteq r & \text{Definition 11.7.1} \\
& \Leftrightarrow p{\bullet}q \subseteq r & \text{Fact} \\
& \Leftrightarrow r \in (p{\bullet}q)^{\bullet} & \text{Definition 11.7.1}
\end{array}
$$

(4) We reason as follows:

$$
\begin{array}{rll}
r \in q^{\bullet}{-\!\!\bullet}p^{\bullet} & \Leftrightarrow \forall q'{\in}q^{\bullet}. r{\circ}q' \subseteq p^{\bullet} & \text{Proposition 9.2.6} \\
& \Leftrightarrow \forall q'. q \subseteq q' \Rightarrow (r{\bullet}q')^{\bullet} \subseteq p^{\bullet} & \text{Definition 11.7.1} \\
& \Leftrightarrow \forall q'. q \subseteq q' \Rightarrow p \subseteq r{\bullet}q' & \text{Definition 11.7.1} \\
& \Leftrightarrow p \subseteq r{\bullet}q & \text{Fact} \\
& \Leftrightarrow q{-\!\!\bullet}p \subseteq r & \text{Lemma 11.3.4} \\
& \Leftrightarrow r \in (q{-\!\!\bullet}p)^{\bullet} & \text{Definition 11.7.1}
\end{array}
$$

□

Corollary 11.7.12 resembles Lemma 9.4.9 and is proved similarly:

Corollary 11.7.12. *Suppose $p, q \in |points_{\Pi}|$ and $u \in |\mho|$. Then $\bullet$ and ${-\!\!\bullet}$ validate axioms $(\sigma\bullet)$ and $(\sigma{-\!\!\bullet})$ from Figure 3:*

$$
\begin{aligned}
(p^{\bullet}{\bullet}q^{\bullet})[a{\mapsto}u] &= p^{\bullet}[a{\mapsto}u]{\bullet}q^{\bullet}[a{\mapsto}u] \\
(b{\Uparrow}^{\bullet}_{\Pi}{-\!\!\bullet}p^{\bullet})[a{\mapsto}u] &= b{\Uparrow}^{\bullet}_{\Pi}{-\!\!\bullet}p^{\bullet}[a{\mapsto}u]
\end{aligned}
$$

Proof. We reason as follows:

$$
\begin{array}{rll}
(p^{\bullet}{\bullet}q^{\bullet})[a{\mapsto}u] & = (p{\bullet}q)^{\bullet}[a{\mapsto}u] & \text{Part 3 of Proposition 11.7.11} \\
& = ((p{\bullet}q)[a{\mapsto}u])^{\bullet} & \text{Part 1 of Theorem 11.7.8} \\
& = (p[a{\mapsto}u]{\bullet}q[a{\mapsto}u])^{\bullet} & \text{Corollary 11.6.3} \\
& = (p[a{\mapsto}u])^{\bullet}{\bullet}(q[a{\mapsto}u])^{\bullet} & \text{Part 3 of Proposition 11.7.11} \\
& = p^{\bullet}[a{\mapsto}u]{\bullet}q^{\bullet}[a{\mapsto}u] & \text{Part 1 of Theorem 11.7.8} \\[1ex]
(b{\Uparrow}^{\bullet}_{\Pi}{-\!\!\bullet}p^{\bullet})[a{\mapsto}u] & = (b{\Uparrow}_{\Pi}{-\!\!\bullet}p)^{\bullet}[a{\mapsto}u] & \text{Part 4 of Proposition 11.7.11} \\
& = ((b{\Uparrow}_{\Pi}{-\!\!\bullet}p)[a{\mapsto}u])^{\bullet} & \text{Part 1 of Theorem 11.7.8} \\
& = (b{\Uparrow}_{\Pi}{-\!\!\bullet}(p[a{\mapsto}u]))^{\bullet} & \text{Proposition 11.6.4} \\
& = b{\Uparrow}^{\bullet}_{\Pi}{-\!\!\bullet}(p[a{\mapsto}u])^{\bullet} & \text{Part 4 of Proposition 11.7.11} \\
& = b{\Uparrow}^{\bullet}_{\Pi}{-\!\!\bullet}p^{\bullet}[a{\mapsto}u] & \text{Part 1 of Theorem 11.7.8}
\end{array}
$$

□

Corollary 11.7.13. *If $s', s' \in |\mho|$ then $s'{\Uparrow}^{\bullet}_{\Pi}{\bullet}s{\Uparrow}^{\bullet}_{\Pi} = (s's){\Uparrow}^{\bullet}_{\Pi}$ and $\boldsymbol{\lambda}a.(s{\Uparrow}^{\bullet}_{\Pi}) = (\lambda a.s){\Uparrow}^{\bullet}_{\Pi}$.*

Proof. We reason as follows:

$$
\begin{array}{rll}
s'{\Uparrow}^{\bullet}_{\Pi}{\bullet}s{\Uparrow}^{\bullet}_{\Pi} & = (s'{\Uparrow}_{\Pi}{\bullet}s{\Uparrow}_{\Pi})^{\bullet} & \text{Part 3 of Prop 11.7.11} \\
& = (s's){\Uparrow}^{\bullet}_{\Pi} & \text{Lemma 11.3.8} \\[1ex]
\boldsymbol{\lambda}a.(s{\Uparrow}^{\bullet}_{\Pi}) & = \boldsymbol{\forall}a.(a{\Uparrow}^{\bullet}_{\Pi}{-\!\!\bullet}s{\Uparrow}^{\bullet}_{\Pi}) & \text{Notation 10.2.1} \\
& = \boldsymbol{\forall}a.(a{\Uparrow}_{\Pi}{-\!\!\bullet}s{\Uparrow}_{\Pi})^{\bullet} & \text{Part 4 of Prop 11.7.11} \\
& = (\boldsymbol{\forall}a.(a{\Uparrow}_{\Pi}{-\!\!\bullet}s{\Uparrow}_{\Pi}))^{\bullet} & \text{Part 2 of Prop 11.7.11} \\
& = (\lambda a.s){\Uparrow}^{\bullet}_{\Pi} & \text{Proposition 11.3.9}
\end{array}
$$

□

11.8. A topology

11.8.1. Giving $points_{\Pi}$ a topology

Definition 11.8.1. Make $points_{\Pi}$ into a nominal spectral space with $\circ$ (Definition 9.2.9):

(1) The topology generated under strictly finitely supported unions by $\{p^{\bullet} \mid p \in |points_{\Pi}|\}$.

(2) The combination operation $p\circ q = (p\bullet q)^\bullet$ from Definition 11.7.1.
(3) The pointwise actions from Definitions 3.3.1 (for π and $[u\text{↤}a]$) and 9.2.3 (for $\bullet$ and $-\!\bullet$).
(4) $\partial_{points_\Pi} u = u{\uparrow}^\bullet_\Pi$ for $u \in |\mho|$.

REMARK 11.8.2. So $U \in opens(points_\Pi)$ (meaning that U is open) when there exists some strictly finitely supported $\mathcal{P} \subseteq |points_\Pi|$ (Definition 2.4.2) with $U = \bigcup\{p^\bullet \mid p \in \mathcal{P}\}$.

PROPOSITION 11.8.3. *Definition 11.8.1 does indeed determine a nominal $\sigma\circ$-topological space in the sense of Definition 9.2.2.*

Proof. We start with the conditions from Definition 7.1.1. By Corollary 11.2.5 $(|points_\Pi|, \cdot, \mho, \text{ծ})$ is an ծ-algebra.

Consider $U, V \in opens(points_\Pi)$. As noted in Remark 11.8.2 $U = \bigcup\{p^\bullet \mid p \in \mathcal{P}\}$ and $V = \bigcup\{q^\bullet \mid q \in \mathcal{Q}\}$ for some strictly finitely supported $\mathcal{P}, \mathcal{Q} \subseteq |points_\Pi|$. We check the first three conditions spelled out in Definition 7.1.1:

(i) *U has finite support.* From Theorem 2.3.1.
(ii) *$a\#U$ implies $U[a\mapsto u] = U$.* Suppose $a\#U$. By Lemma 2.4.3 and Theorem 2.3.1 $a\#p^\bullet$ for every $p \in \mathcal{P}$. By Corollary 11.7.10 $p^\bullet[a\mapsto u] = p^\bullet$ for each $p \in \mathcal{P}$. The result follows by Lemma 9.2.11.
(iii) *$b\#U$ implies $U[a\mapsto u] = ((b\ a)\cdot U)[b\mapsto u]$.* Much as the previous case, using Corollary 11.7.10.

We now reason as follows:

(1) *If U is open then so are $\pi\cdot U$ and $U[a\mapsto u]$.* Using Theorem 2.3.1 and Corollary 11.7.9 we have that:
$$\begin{aligned} \pi\cdot U &= \bigcup\{(\pi\cdot p)^\bullet \mid p \in \mathcal{P}\} \qquad \text{and} \\ U[a\mapsto u] &= \bigcup\{(p[a\mapsto u])^\bullet \mid p \in \mathcal{P}\}. \end{aligned}$$
and using Theorem 2.3.1 $\{\pi\cdot p \mid p \in \mathcal{P}\}$ and $\{p[a\mapsto u] \mid p \in \mathcal{P}\}$ are strictly finitely supported. The result follows.
(2) *$\varnothing$ and $|points_\Pi|$ are open.* $\varnothing$ (the empty set of points) is open by construction of the topology, and $|points_\Pi| = \bigcup\{\varnothing^\bullet\}$ is open—we noted in Remark 11.1.4 that $\varnothing$ the empty set of phrases is a point. Then $\varnothing^\bullet$ is the set of all points and $\{\varnothing^\bullet\}$ is strictly finitely supported (by $\varnothing$ the empty set of atoms).
(3) *If U and V are open then so are $U\cap V$ and $U\cup V$.* It is a fact of sets that $U\cap V = \bigcup\{p^\bullet\cap q^\bullet \mid p \in \mathcal{P}, q \in \mathcal{Q}\}$. We use part 1 of Proposition 11.7.11 and Theorem 2.3.1.
For $U\cup V$, it suffices to note that the union of two strictly finitely supported sets is strictly finitely supported.
(4) *If $\mathcal{U}$ is a strictly finitely supported set of open sets then $\bigcup\mathcal{U}$ is open.* Using Corollary 2.4.5.

Finally, we consider Definition 9.2.2 and note that $\circ$ is a combination operator (that $\circ$ is equivariant follows immediately from Theorem 2.3.1). □

In Proposition 11.8.3 we noted that $(|points_\Pi|, \cdot, \mho, \text{ծ})$ is an ծ-algebra by Corollary 11.2.5. We can put together what we have so far and spell out what the corresponding σ-action does concretely to open sets:

PROPOSITION 11.8.4. *Suppose $X \in opens(points_\Pi)$, so that by construction in Definition 11.8.1 $X = \bigcup\{p^\bullet \mid p \in \mathcal{P}\}$ for some strictly finitely supported set of points $\mathcal{P} \subseteq |points_\Pi|$. Suppose $u \in |\mho|$. Then we can describe $X[a\mapsto u]$ equivalently as follows:*
$$\begin{aligned} X[a\mapsto u] &= \{p \mid \text{И}c.p[u\text{↤}c] \in (c\ a)\cdot X\} && \textit{Definition 3.4.1} \\ &= \bigcup\{p^\bullet[a\mapsto u] \mid p \in \mathcal{P}\} && \textit{Lemma 5.1.1} \\ &= \bigcup\{(p[a\mapsto u])^\bullet \mid p \in \mathcal{P}\} && \textit{Theorem 11.7.8} \\ &= \bigcup\{(\text{и}a.(p[a{:=}u]))^\bullet \mid p \in \mathcal{P}\} && \textit{Lemma 11.4.9} \end{aligned}$$

11.8.2. The compact open sets of $points_\Pi$

LEMMA 11.8.5. *If $U \in opens(points_\Pi)$ then*

$$p \in U \quad \textit{if and only if} \quad p^\bullet \subseteq U.$$

Proof. Suppose $p \in U$. By Definition 11.8.1 $U = \bigcup\{q^\bullet \mid q \in \mathcal{Q}\}$ for some strictly finitely supported $\mathcal{Q} \subseteq |points_\Pi|$, and there exists $q \in \mathcal{Q}$ with $p \in q^\bullet$. By Lemma 11.7.6, $p^\bullet \subseteq q^\bullet$.
The reverse implication is easy since $p \in p^\bullet$ by construction in Definition 11.7.1. □

LEMMA 11.8.6. *If $p \in |points_\Pi|$ then $p^\bullet$ is compact in $points_\Pi$ with the topology from Definition 11.8.1. In symbols: $p^\bullet \in cpct(points_\Pi)$.*
As a corollary, if $s \in |\mho|$ then $s{\uparrow}^\bullet_\Pi \in cpct(points_\Pi)$.

Proof. Suppose $\mathcal{U}$ covers $p^\bullet$. Since $p \in p^\bullet$, also $p \in U$ for some $U \in \mathcal{U}$. By Lemma 11.8.5 $p^\bullet \subseteq X$ and so $p^\bullet$ is covered by $\{X\}$.
The corollary follows from Lemma 11.2.2. □

LEMMA 11.8.7. *If $U \in opens(points_\Pi)$ is compact then $U = \bigcup\{p^\bullet \mid p \in \mathcal{P}\}$ for some finite $\mathcal{P} \subseteq |points_\Pi|$.*

Proof. Suppose U is compact. By construction $U = \bigcup \mathcal{P}'$ for some strictly finitely supported (but not necessarily finite) $\mathcal{P}' \subseteq |points_\Pi|$. By compactness this has a finite subcover $\{p_1^\bullet, \dots, p_n^\bullet\}$. We take $\mathcal{P} = \{p_1, \dots, p_n\}$. □

REMARK 11.8.8. We have mentioned that the canonical model $points_\Pi$ is not just a replay of $F(\mathfrak{D})$ from the duality proof (Definition 7.2.1). So compare Proposition 7.3.10 (which identifies compacts in $F(\mathfrak{D})$ with sets of points of the form $x^\bullet$) with Lemma 11.8.7 (which identifies compacts in $points_\Pi$ with *finite unions* of sets of the form $p^\bullet$). If $F(\mathfrak{D})$ and $points_\Pi$ were the same, then we would expect Proposition 7.3.10 and Lemma 11.8.7 to also by the same. This is related to issues discussed in Remarks 11.3.2 and 11.5.8.

11.8.3. Interaction of $\bullet$ and $-\!\bullet$ with $\cup$ and $\subseteq$. Lemmas 11.8.9 and 11.8.10 are useful for Corollary 11.8.11:

LEMMA 11.8.9. *Suppose $p, q, q' \in |points_\Pi|$ and $q \subseteq q'$. Then $p{\bullet}q \subseteq p{\bullet}q'$ and $p{\circ}q' \subseteq p{\circ}q$.*

Proof. $p{\bullet}q$ is defined in Definition 11.3.1, and the result follows direct from the definition. By Definition 11.7.1 $p{\circ}q' = \{r{\in}points_\Pi \mid p{\bullet}q' \subseteq r\}$ and $p{\circ}q = \{r{\in}points_\Pi \mid p{\bullet}q \subseteq r\}$. We use Lemma 11.7.6. □

LEMMA 11.8.10. *If $X \in opens(points_\Pi)$ and $p, q \in |points_\Pi|$ then the following conditions are equivalent:*

$$p{\bullet}q \in X \quad \Leftrightarrow \quad (p{\bullet}q)^\bullet \subseteq X \quad \Leftrightarrow \quad p{\circ}q \subseteq X \quad \Leftrightarrow \quad p{\circ}q^\bullet \subseteq X$$

Proof. By Lemma 11.8.5 $p{\bullet}q \in X$ if and only if $(p{\bullet}q)^\bullet \subseteq X$. By Definition 11.7.1 $(p{\bullet}q)^\bullet = p{\circ}q$.
Suppose $p{\circ}q \subseteq X$. Consider any $q' \in q^\bullet$, meaning by Definition 11.7.1 that $q \subseteq q'$. By Lemma 11.8.9 $p{\bullet}q' \subseteq p{\circ}q$. Since q' was arbitrary, it follows from Definition 9.2.4 that $p{\circ}q^\bullet \subseteq X$.
Conversely if $p{\circ}q^\bullet \subseteq X$ then since (from Definition 11.7.1) $q \in q^\bullet$, from Definition 9.2.4 $p{\circ}q \subseteq X$. □

COROLLARY 11.8.11. *If $X \in opens(points_\Pi)$ and $r, q \in |points_\Pi|$ then $r \in (q_1^\bullet \cup \dots \cup q_n^\bullet){-\!\bullet}X$ if and only if $r{\bullet}q_i \in X$ for $1{\leq}i{\leq}n$.*

Proof. By Proposition 9.2.6 $r \in (q_1^\bullet \cup \dots \cup q_n^\bullet){-\!\bullet}X$ if and only if $r{\circ}(q_1^\bullet \cup \dots \cup q_n^\bullet) \subseteq X$. From Definition 9.2.4 this is if and only if $r{\circ}q_i^\bullet \subseteq X$ for $1{\leq}i{\leq}n$. We use Lemma 11.8.10. □

COROLLARY 11.8.12. *$r \in q^\bullet{-\!\bullet}(p_1^\bullet \cup \dots \cup p_n^\bullet)$ if and only if $r \in q^\bullet{-\!\bullet}p_i^\bullet$ for some $1{\leq}i{\leq}n$.*

Proof. By Corollary 11.8.11 $r \in q^\bullet \multimap\!\!\bullet (p_1^\bullet \cup \cdots \cup p_n^\bullet)$ if and only if $r \bullet q \in p_1^\bullet \cup \cdots \cup p_n^\bullet$. It is a fact of sets that this is if and only if $r \bullet q \in p_i^\bullet$ for some $1 \leq i \leq n$. By Corollary 11.8.11 this is if and only if $r \in q^\bullet \multimap\!\!\bullet p_i^\bullet$ for some $1 \leq i \leq n$, as required. □

COROLLARY 11.8.13. *Suppose $\mathcal{P}, \mathcal{Q} \subseteq |points_\Pi|$ are finite. Then*

$$\bigcup\{q^\bullet \mid q \in \mathcal{Q}\} \multimap\!\!\bullet \bigcup\{p^\bullet \mid p \in \mathcal{P}\} = \bigcap\{\bigcup\{(q \multimap\!\!\bullet p)^\bullet \mid p \in \mathcal{Q}\} \mid q \in \mathcal{Q}\}.$$

Proof. We combine Corollaries 11.8.11 and 11.8.12 with part 4 of Proposition 11.7.11. □

11.8.4. Interaction of $\bigcap^{\#a}$ with unions. Remarkably, $\bigcap^{\#a}$ commutes with certain unions. This is Lemma 11.8.14. The property is not valid in general in inDi$\forall_\bullet$; indeed this is not generally true in logic: $\forall x.(\phi \vee \psi)$ is not normally logically equivalent to $(\forall x.\phi) \vee (\forall x.\psi)$. But, it holds in the canonical model $points_\Pi$:

LEMMA 11.8.14. $\bigcap^{\#a} \bigcup_i p_i^\bullet = \bigcup_i \bigcap^{\#a} p_i^\bullet$.

Proof. Suppose $q \in \bigcap^{\#a} \bigcup_i p_i^\bullet$. Using Lemma 5.2.5 rename to assume without loss of generality that $a \# q$.

Then by Definition 5.2.1 and Proposition 3.4.2 $q \in \bigcap^{\#a} \bigcup_i p_i^\bullet$ is if and only if $q[u \leftarrow\!\!\dashv a] \in \bigcup_i p_i^\bullet$ for every $u \in |\mho|$. Choose fresh b (so $b \# q, p_1, \ldots, p_n$). Then for some $1 \leq i \leq n$ we have that $q[b \leftarrow\!\!\dashv a] \in p_i^\bullet$. Therefore $q \in (b\ a) \cdot p_i^\bullet$ and so by Lemma 5.2.5 $q \in \bigcap^{\#b} (b\ a) \cdot p_i^\bullet = \bigcap^{\#a} p_i^\bullet$.

Conversely, it is easy to prove that $\bigcap^{\#a} p_i^\bullet \subseteq \bigcap^{\#a} \bigcup_i p_i^\bullet$, either using Lemma 4.1.7 and Corollary 5.2.7 since $p_i^\bullet \subseteq \bigcup_i p_i^\bullet$—or by an easy direct calculation from Definition 5.2.1 and part 4 of Lemma 5.1.1. □

11.8.5. Proof that $points_\Pi$ is coherent and sober. We saw in Proposition 11.8.3 that $points_\Pi$ is a nominal $\sigma\circ$-topological space in the sense of Definition 9.2.2. We now show that it is coherent (Definitions 7.4.1 and 9.2.7) and sober (Definition 7.7.2).

LEMMA 11.8.15. *Suppose U and V are compact in $points_\Pi$ and suppose $u \in |\mho|$. Then:*

— *$|points_\Pi|$ is compact.*
— *$U \cap V$ is compact.*
— *$U[a \mapsto u]$ is compact.*

Proof. By Lemma 11.8.7 we may assume $U = \bigcup\{p^\bullet \mid p \in \mathcal{P}\}$ and $V = \bigcup\{q^\bullet \mid q \in \mathcal{Q}\}$ for finite $\mathcal{P}, \mathcal{Q} \subseteq |points_\Pi|$. We consider each part in turn:

— $|points_\Pi| = \varnothing^\bullet$ (which is a point, as noted in Remark 11.1.4). We use Lemma 11.8.6.
— Using part 1 of Proposition 11.7.11 $U \cap V = \bigcup\{(p \wedge q)^\bullet \mid p \in \mathcal{P}, q \in \mathcal{Q}\}$. By Lemma 11.8.6 each $(p \wedge q)^\bullet$ is compact, and a finite union of compact sets is compact.
— By Lemma 5.1.1 and Theorem 11.7.8 $U[a \mapsto u] = \bigcup\{(p[a \mapsto u])^\bullet \mid p \in \mathcal{P}\}$. By Lemma 11.8.6 each $(p[a \mapsto u])^\bullet$ is compact, and a finite union of compact sets is compact.

□

LEMMA 11.8.16. *Suppose U is compact in $points_\Pi$. Then continuing Lemma 11.8.15:*

— *$\bigcap^{\#a} U$ is compact.*

Proof. By Lemma 11.8.7 we may assume $U = \bigcup\{p^\bullet \mid p \in \mathcal{P}\}$ for finite $\mathcal{P} \subseteq |points_\Pi|$. By Lemma 11.8.14 and by part 2 of Proposition 11.7.11 $\bigcap^{\#a} U = \bigcup\{(\forall a.p)^\bullet \mid p \in \mathcal{P}\}$. By Lemma 11.8.6 each $(\forall a.p)^\bullet$ is compact, and a finite union of compact sets is compact. □

PROPOSITION 11.8.17. *Suppose U and V are compact in $points_\Pi$. Then continuing Lemma 11.8.16:*

— *$U \bullet V$ is compact.*
— *$V \multimap\!\!\bullet U$ is compact.*

Proof. By Lemma 11.8.7 we may assume $U = \bigcup\{p^\bullet \mid p \in \mathcal{P}\}$ and $V = \bigcup\{q^\bullet \mid q \in \mathcal{Q}\}$ for finite $\mathcal{P}, \mathcal{Q} \subseteq |points_\Pi|$. We consider each part in turn:

— By Lemma 9.2.11 and part 3 of Proposition 11.7.11 $U{\bullet}V = \bigcup\{(p{\bullet}q)^\bullet \mid p \in \mathcal{P},\ q \in \mathcal{Q}\}$. By Lemma 11.8.6 each $(p{\bullet}q)^\bullet$ is compact, and a finite union of compact sets is compact.
— By Corollary 11.8.13 $V{-\!\!\bullet}U = \bigcap_{q\in\mathcal{Q}} \bigcup_{p\in\mathcal{P}} (q{-\!\!\bullet}p)^\bullet$. By Lemma 11.8.6 each $(q{-\!\!\bullet}p)^\bullet$ is compact; by Lemma 11.8.15 a finite intersection of compact sets is compact; and a finite unions of compact sets is compact.

$\square$

Corollary 11.8.18. *$points_\Pi$ is a coherent nominal $\sigma\circ$-topological space.*

Proof. $points_\Pi$ is a nominal $\sigma\circ$-topological space by Proposition 11.8.3. It remains to check the additional coherence conditions of Definitions 7.4.1 and 9.2.7.

We check the conditions of Definition 7.4.1. Suppose U and V are compact in $points_\Pi$. By Proposition 11.8.17 $U[a{\mapsto}u]$, $|points_\Pi|$, and $U \cap V$ are compact. By Lemma 11.8.14 $\bigcap^{\#a} U$ is compact.

By construction in Definition 11.8.1 every open set is a strictly finitely supported union of compact open sets (of the form $p^\bullet$ for $p \in |points_\Pi|$).

We check the conditions of Definition 9.2.7. Suppose U and V are compact. By Proposition 11.8.17 $U{\bullet}V$ and $V{-\!\!\bullet}U$ are compact. Conditions 6 and 7 of Definition 9.2.7 also hold (validity of $(\sigma\bullet)$ and $(\sigma{-\!\!\bullet})$):

$$\begin{aligned}
(U{\bullet}V)[a{\mapsto}u] &= \bigcup\{(p^\bullet{\bullet}q^\bullet)[a{\mapsto}u] \mid p{\in}\mathcal{P},\ q{\in}\mathcal{Q}\} && \text{Lemmas 9.2.11 \& 5.1.1}\\
&= \bigcup\{p^\bullet[a{\mapsto}u]{\bullet}q^\bullet[a{\mapsto}u] \mid p{\in}\mathcal{P},\ q{\in}\mathcal{Q}\} && \text{Corollary 11.7.12}\\
&= \bigcup\{p^\bullet[a{\mapsto}u] \mid p{\in}\mathcal{P}\}{\bullet}\{q^\bullet[a{\mapsto}u] \mid q{\in}\mathcal{Q}\} && \text{Lemma 9.2.11}\\
&= U[a{\mapsto}u]{\bullet}V[a{\mapsto}u] && \text{Lemmas 9.2.11 \& 5.1.1}
\end{aligned}$$

$$\begin{aligned}
(\partial_{points_\Pi} b{-\!\!\bullet}U)[a{\mapsto}u] &= \left(\textstyle\bigcup_{p\in\mathcal{P}} b^\bullet{-\!\!\bullet}p^\bullet\right)[a{\mapsto}u] && \text{Def 11.8.1 \& Cor 11.8.13}\\
&= \textstyle\bigcup_{p\in\mathcal{P}}((b^\bullet{-\!\!\bullet}p^\bullet)[a{\mapsto}u]) && \text{Lemma 5.1.1}\\
&= \textstyle\bigcup_{p\in\mathcal{P}} b^\bullet{-\!\!\bullet}p^\bullet[a{\mapsto}u] && \text{Corollary 11.7.12}\\
&= \partial_{points_\Pi} b{-\!\!\bullet}(U[a{\mapsto}u]) && \text{Cor 11.8.13 \& Lem 5.1.1}
\end{aligned}$$

$\square$

Lemma 11.8.19. *$points_\Pi$ is sober (Definition 7.7.2).*

Proof. It suffices to observe that p is uniquely identified by the set of open (indeed, compact) sets $\{s{\uparrow}^\bullet_\Pi \mid s \in p\}$. $\square$

Theorem 11.8.20. *$points_\Pi$ from Definition 11.8.1 is indeed a nominal spectral space with $\circ$.*

Proof. By Proposition 11.8.3 $points_\Pi$ is a nominal $\sigma\circ$-topological space. By Corollary 11.8.18 it is coherent and by Lemma 11.8.19 it is sober. It remains to check that it is impredicative; that is, that $\partial_{points_\Pi} = \text{-}^\bullet$ is a morphism of σ-algebras (Definition 4.4.4) from $\mho$ to $cpct(points_\Pi)$. This is just Theorem 11.7.8. $\square$

11.9. Logical properties of the topology, and completeness

Recall that at the start of this section we fixed an idiom $\mho$ (Definition 10.3.1) and a λ-reduction theory Π over $\mho$ (Definition 10.3.7).

Remark 11.9.1. By Corollary 11.8.18 $points_\Pi$ is coherent. Thus from Definition 7.6.1 and Theorem 7.6.2 we have that $G(points_\Pi)$ is an nominal distributive lattice with $\forall$; it consists of compact open sets in $points_\Pi$ ordered by subset inclusion. By Lemma 11.8.7, each compact open set is a finite union of sets of the form $p^\bullet$ for $p \in |points_\Pi|$.

NOTATION 11.9.2. We call $G(points_\Pi)$ the **canonical model**.

We now set about proving Theorem 11.9.5, which uses $G(points_\Pi)$ to prove completeness—the converse direction to soundness from Theorem 10.4.7.

Recall from Definition 10.4.1 the definition of $[\![s]\!]^{G(points_\Pi)}$ and recall from Definition 10.1.2 that LmTm is the set of λ-terms. Suppose $\mho = \mathsf{LmTm}$.

LEMMA 11.9.3. *$[\![s]\!]^{G(points_\Pi)} = s{\Uparrow}^{\bullet}_{\Pi}$ and as a corollary $[\![s]\!]^{G(points_\Pi)} \in |G(points_\Pi)|$.*

Proof. By a routine induction on λ-terms:

— $[\![a]\!]^{G(points_\Pi)} \overset{\text{Def 10.4.1}}{=} \partial_{G(points_\Pi)} a \overset{\text{Def 11.8.1}}{=} a{\Uparrow}^{\bullet}_{\Pi}$.

— $[\![s's]\!]^{G(points_\Pi)} \overset{\text{Def 10.4.1}}{=} [\![s']\!]^{G(points_\Pi)} \bullet [\![s]\!]^{G(points_\Pi)} \overset{\text{ind hyp}}{=} s'{\Uparrow}^{\bullet}_{\Pi} \bullet s{\Uparrow}^{\bullet}_{\Pi} \overset{\text{Cor 11.7.13}}{=} (s's){\Uparrow}^{\bullet}_{\Pi}$.

— $[\![\lambda a.s]\!]^{G(points_\Pi)} \overset{\text{Def 10.4.1}}{=} \boldsymbol{\lambda} a.[\![s]\!]^{G(points_\Pi)} \overset{\text{ind hyp}}{=} \boldsymbol{\lambda} a.(s{\Uparrow}^{\bullet}_{\Pi}) \overset{\text{Cor 11.7.13}}{=} (\lambda a.s){\Uparrow}^{\bullet}_{\Pi}$.

The corollary follows from Lemma 11.8.6, since $|G(points_\Pi)|$ is by definition the set of compact open sets of $points_\Pi$. ☐

Recall from Notation 10.3.8 and Definition 10.4.1 the notations $\Pi \vdash s \to t$ and $G(points_\Pi) \vDash \Pi$.

PROPOSITION 11.9.4. $G(points_\Pi) \vDash \Pi$.

Proof. Unpacking Definition 10.4.1, we must show that $(s \to t) \in \Pi$ implies $[\![s]\!]^{G(points_\Pi)} \leq [\![t]\!]^{G(points_\Pi)}$, where $\leq$ means $\subseteq$.

So suppose $(s \to t) \in \Pi$. By Notation 10.3.8 this means precisely $s \twoheadrightarrow_\Pi t$ and it follows by Lemma 11.7.4 that $s{\Uparrow}^{\bullet}_{\Pi} \subseteq t{\Uparrow}^{\bullet}_{\Pi}$, so by Lemma 11.9.3 $[\![s]\!]^{G(points_\Pi)} \subseteq [\![t]\!]^{G(points_\Pi)}$ as required. ☐

THEOREM 11.9.5 (Completeness). *$\Pi \vdash s \to t$ (or equivalently: $s \twoheadrightarrow_\Pi t$) if and only if $\Pi \vDash s \leq t$.*

Proof. The left-to-right implication is Theorem 10.4.7. Now suppose $\Pi \vDash s \leq t$. By Proposition 11.9.4 $points_\Pi \vDash \Pi$ so $[\![s]\!]^{G(points_\Pi)} \subseteq [\![t]\!]^{G(points_\Pi)}$. By Lemma 11.9.3 $[\![s]\!]^{G(points_\Pi)} = s{\Uparrow}^{\bullet}_{\Pi}$ and $[\![t]\!]^{G(points_\Pi)} = t{\Uparrow}^{\bullet}_{\Pi}$. By Lemma 11.7.4 $s \twoheadrightarrow_\Pi t$. ☐

11.10. Interlude: an interesting disconnect

The duality theorem from Theorem 9.6.6 is more general than the completeness theorem needs it to be. The completeness result of Theorem 11.9.5 is based on $points_\Pi \in \mathsf{inSpect}\forall_\bullet$. Although $points_\Pi$ is a spectral space (Theorem 11.8.20) it has more structure too: for instance it is replete (Definition 10.4.3) and has an existential quantifier, as we note later in Definition B.2.1.

Could we obtain a more specific duality result for structures that have more of the structure apparent in $points_\Pi$?

We probably could. However, we do not do it in this paper. $\mathsf{inDi}\forall_\bullet/\mathsf{inSpect}\forall_\bullet$ have the minimal structure we need to interpret the λ-calculus and to carry out a filter-based duality proof. The less structure we impose, the more general our duality result,[40] and we have more representations.

But in the completeness result we are happy if the canonical model has more structure, since it suggests more programming and reasoning constructs; an existential quantifier, for example, suggests that the ambient meta-logic implicit in $points_\Pi$ permits unconstrained search. We do not care about any other structures because we have built one *particular* concrete structure and having built it, we want to obtain as many bells and whistles from it for free as possible.[41]

So on the one hand we have a world where less structure is good, because fewer assumptions means stronger theorems that are valid for a larger class of entities (provided we can still build the things we

[40] Broadly speaking, within a given class of structures, the less structure we assume the more challenging the duality result is to prove. If duality theory were a competitive sport then it would be like golf: the lower your score the better your game.

[41] So canonical models are like tennis: more points is better.

want to in those entities, i.e. interpret the λ-calculus, which we can), and on the other hand we have a world where more structure is good, because it gives us more tools to actually do things.

The apparent disconnect comes from a difference between two styles, each of which is optimised for its own purpose.

The general trend in this paper is a progression from the abstract and general, like $\mathsf{inSpect}\forall_{\bullet}$, to the relatively more concrete and specific, like $points_{\Pi}$.

12. CONCLUSIONS

The semantics of this paper has the moderately unusual feature of being *absolute*, meaning that variables are interpreted directly in the denotation and there is no (Tarski-style) valuation.

The reader may find this takes some getting used to, but it is actually simple and natural.

What corresponds to valuations is the σ-action, which allows us to take some x and 'evaluate' a to u in x by forming $x[a{\mapsto}u]$. This is an abstract nominal algebraic property of x; it is characterised by axioms and we do not necessarily have access to the internal structure of x.

However, we can certainly build concrete σ-algebras if we want to: Two examples are λ-term syntax LmTm from Definition 10.1.1 and the canonical model $points_{\Pi}$ from Subsection 11.9. Another example is how we move from $\amalg$-algebra structure to σ-algebra structure (and back) using nominal powersets (Definition 3.4.1). Nominal powersets have intersections, unions, and complements, and by combining all of these things we can interpret $\forall$ (Definition 5.2.1).

In fact, it turns out that with a little more effort and just a bit more structure we can interpret application and λ too. This brings us on to another unusual feature of our topological semantics: it is *purely* sets-based.

Algebraic (dually: topological) semantics for the λ-calculus exist, but our semantics is this in a different and stronger sense than usual, because *everything* is interpreted algebraically (dually: topologically), including variables, substitution, and λ-abstraction.

This paper gives a panoramic view of the interaction between nominal foundations and the λ-calculus. This gives us something that shorter papers might not do so well: a feel for the overall point of view, and how the parts of the puzzle fit together.

12.1. Related work

12.1.1. Algebraic semantics. Algebraic semantics for logics or calculi with binding include polyadic algebras [Hal06, Part II], cylindric algebras [HMT85], and Lambda Abstraction Algebras [Sal00]. As far as we know, what is done in this paper has not been done in any of these (but see below).

We can suggest technical reasons for this. Consider for instance the treatment of substitution in this paper.

For us substitution exists independently from β-reduction—this is the notion of σ-algebra from Subsection 3.1. This is important for our constructions to work. For instance, Subsections 3.3 and 3.4 do not assume λ and application, they only assume σ and $\amalg$. This is reflected in commutation results like Lemma 6.4.1 and Theorem 11.7.8.

The commutations for λ are later, and much harder: Lemmas 9.4.9 and 9.4.10 for $\mathsf{inDi}\forall_{\bullet}$ and Corollary 11.7.13 for $points_{\Pi}$.

It not obvious how substitution *on its own* could be axiomatised without permutations and freshness side-conditions, i.e. without nominal algebra. LAAs do not do this, neither do cylindric algebras. Polyadic algebras *assume* a monoid of substitutions. This is tantalisingly close to finite permutations, but without their invertibility.

By enriching the foundation with names and binding, nominal techniques allow us to express richer algebraic structures—for instance substitution, as was done in [GM06a; GM08a]. We exploited that fact in this paper to break constructions up into more manageable parts: we split λ into $\forall$ and $\multimap\!\bullet$ (Notation 10.2.1), β-reduction into $\bullet$, $\multimap\!\bullet$, and σ (Proposition 10.2.4), and then σ (substitution) itself into permutation and freshness (Definitions 3.1.4 or 3.4.1).

Representation theorems exist for cylindric algebras; for instance [Mon61] gives a representation theorem for cylindric algebras, and [PS95] gives one for LAAs. In both cases, an algebra is represented concretely as a set of valuations on the variables (on the indexes; the things that correspond to atoms in this paper).

This is typical. Representation theorems for cylindric-algebra-style systems do all seem to use something corresponding to a set of valuations. It works, but it is a soundness and completeness proof with respect to Tarski-style semantics. There is no duality result.

The only duality results we know of for logic were undertaken by Forssel [For07] and by the first author [Gab11b]. See [Gab11b] for a comparison of the two.

A *Stone Representation* has been given for Lambda Abstraction Algebras. This is a factorisation result in the style of the HSP theorem (also known as Birkhoff's theorem): every LAA can be factored as a product of irreducible LAAs. The factorisations are identified by *central elements* of the algebra [MS10]. The LAA is never represented as anything resembling a Stone space, and there is no duality.

In passing, we note that the HSP part of the LAA result also follows in this paper (for $\mathsf{inDi}\forall_{\bullet}$) off-the-shelf, by the *nominal* HSPA theorem [Gab09a; Gab13]. In other words, we get some of [MS10] for free, just by virtue of being nominal and using nominal algebra. The HSPA factorisation is slightly better than the HSP factorisation (because it has an A in it: for atoms-abstraction). Investigating any extra power this gives what that might mean for (nominal) LAAs, is an open problem.

We mention also [KP10]. This is an attempt to encode what makes nominal techniques work using many-sorted universal algebra. Equivariance, however, gets lost in the translation; a similar phenomenon was noted in [DG12] translating permissive-nominal logic (a first-order generalisation of nominal algebra) to higher-order logic.

12.1.2. Absolute semantics. Absolute semantics have appeared before. Lambda-abstraction algebras (for the λ-calculus) and cylindric algebras and polyadic algebras (for first-order logic) are absolute. Selinger made a case for using absolute semantics for the λ-calculus in [Sel02] (see Subsection 2.2); a line of thought echoed by the first author with Mulligan in [GM11].

Yet absolute semantics have not caught on. We are inclined to believe that this is because the mathematical foundations to support it were not in place before, but they are now. Now that we have nominal techniques we can make a lot of things work that would not work before.

Without nominal techniques things we use repeatedly in the current paper, like finite support, freshness side-conditions, equivariance, the И-quantifier, and even α-equivalence, become challenging in various technical fiddly ways, and even the statements of some properties become practically impossible to even write out.

Concretely, let us imagine how we might set about rendering condition 4 of Definition 6.1.1, or Definition 11.4.1, or even the clause for $\forall a.p$ in Definition 11.3.1, if we did not have finite support, freshness, and the И-quantifier. We would probably have to invent them first.

We also mention Kit Fine's *arbitrary objects* [Fin85] as an instance of a similar impulse towards absolute semantics, coming from philosophy. This comes from philosophy.

There is a precedent for this paper in the first author's work; indeed this paper is based on them. The nominal semantics and duality results for first-order logic in [Gab11b] and [Gab12] are absolute, and are very much in the style and research programme of this paper.

Nominal algebra has helped us to reduce mathematical overhead and to simplify some technical manipulations that are otherwise all too easy to get bogged down in. This is just what any good mathematical toolbox or foundation should do.

12.1.3. η-expansion. In Proposition 10.2.4 we saw β-reduction and η-expansion appear spontaneously as corollaries of adjoint properties. So our notion of λ-reduction theory is more general than an extensional λ-equality theory because reductions can go one way and not the other, but it is also more specific than just any set of reductions because it must contain η-expansion.

The reader used to seeing η as a contraction rule in rewrite systems might be interested in a thread of publications by Barry Jay and Neil Ghani, which argues in favour of η-expansion from the point of view of rewriting, for better confluence and other properties which they list. See [JG95] and [Gha97].

For us too, expansion rather than contraction seems to be the natural primitive.

On the basis of preliminary calculations in unpublished notes, we believe that we can remove η-expansion at some cost in complexity in the models. We do this by considering *two* application operations $\bullet'$ and $\bullet$; one intensional and one extensional. This is future work.

12.1.4. Previous treatment of λ-calculus by the authors. In [GM06a; GM06b] the first author and Mathijssen developed *nominal algebra* and axiomatised substitution and first-order logic, with completeness proofs. Journal versions are [GM08a; GM08c].

An axiomatisation, again with completeness proofs, for the λ-calculus followed in [GM08b; GM10].

So the σ-axioms which appear in Figure 1 are taken from [GM06a], and the axioms for β and η are descended from [GM08b; GM10].

In [Gab11b] we applied duality theory to in nominal sets to the axiomatisation of [GM06b; GM08c]. The main conceptual challenge (aside from the inherent difficulty of duality proofs) was to invent $\amalg$-algebras.[42] The $\amalg$-axioms of Figure 1 are from [Gab11b]. We have taken this further in [Gab12].

This paper carries out a similar project to [Gab11b], but for the λ-calculus. This has been a tougher target than first-order logic, which is unsurprising. The main conceptual difficulty of this paper over the previous work is the treatment of application and λ using adjoints and the logical quantifier. The ideas for this are from [GG10] (see Figure 2, where $-\!\bullet$ is written $\triangleright$). The similarity with [GG10] is somewhat hidden just because it was written in a 'modal logic' style. That style has been replaced in this paper by the nominal foundations.

In summary, and at least in principle, this paper just combines [GM06a], [GM08b], and [GG10] with [Gab11b]. (What could be simpler or more natural?)

12.1.5. No conflict with topological incompleteness results. The best-known models of the untyped λ-calculus are Scott's domain models and generalisations: graph semantics; filter semantics; stable semantics; strongly stable semantics; and so on. An excellent discussion with references—an annotated bibliography and survey, in fact—appears in [Sal01] between Theorems 4.5 and 4.6.

These are all ordered structures, and this is key, since the idea is to reduce the function space using continuity conditions.

These semantics are all incomplete. That is, domains-based denotational semantics proved the λ-calculus consistent, but results like [Sal03, Theorems 3.5 and 4.9] proved that this is not the whole story: see also [Sal01].[43]

The reader familiar with this literature and who has seen e.g. Theorem 3.5 of [Sal03] might be puzzled by Theorem 11.9.5: the former states that no semantics in terms of partially-ordered models with a bottom element can be complete, whereas the latter claims to prove completeness for a semantics based on $\mathsf{inDi}\forall_\bullet$, and an object of $\mathsf{inDi}\forall_\bullet$ is a lattice and has a bottom element $\bot$.

However, nothing insists that $\bot$ should be a program. That is, in the notation of Notation 4.5.3, it is perfectly possible that $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ and $\bot \notin \partial\mathcal{D}$.

This illustrates that $\mathcal{D}$ is a logical structure—its dual is topological—and just a *subset* $\partial\mathcal{D}$ is deemed to be 'computational'. The models of the λ-calculus live in $\partial\mathcal{D} \subseteq \mathcal{D}$, and $\partial\mathcal{D}$ need not be closed under meets or joins.

The formal sense in which this is intended is just that programs are the things that can be substituted for by the σ-action; so intuitively atoms in $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ 'range over' programs. In the light of this reading of the definitions, Definition 10.4.3 calls $\mathcal{D}$ *replete* when its programs are Turing complete.

It remains to discover whether there exists a λ-equality theory such that if $\mathcal{D} \in \mathsf{inDi}\forall_\bullet$ is a model of that theory then it can have no non-trivial order on its *programs* (so if $x, y \in |\partial\mathcal{D}|$ then if $x \leq y$ then $y \leq x$).

[42] This took a couple of years: once the first author understood that for a duality result, a *dual* to σ was needed, the paper was easy to write. At least, for a certain highly technical value of 'easy'.

[43] Page 2 of [Sal03] includes a brief but comprehensive history of such results. The first incompleteness result was given in [HDR92] for the continuous semantics (Scott's construction). This was followed by several generalisations. Salibra's treatment has the benefit of covering a range of semantics in a uniform way.

12.1.6. Sheaves. We impose a topology on a set to reduce the size of the function-space by restricting to continuous functions. Sheaves do much the same thing, but in more generality.

Nominal sets form a category which admits a sheaf presentation (a discussion specific to nominal techniques is in [Gab11a]). Simplifying a little, this amounts to observing that equivariance (commuting with the permutation action) can be represented as a generalised 'continuity' condition. There is no need to stop there. We could try to make 'continuity' represent, for instance, *compatibility* conditions such as $(\sigma\bullet)$ from Figure 3.

This is what is done by the *Topological representation of the λ-calculus* considered in [Awo00]. Examining equation (15) of the paper we see that, essentially, an open set is a set of substitution instances of evaluations from variables to terms. (The calculations are given only for the simply-typed λ-calculus.) Continuity ensures that function application commutes with substitution, i.e. $(\sigma\bullet)$.

The closest thing to the construction of [Awo00] is the construction of $points_{\Pi}$ in Definition 11.1.3.

Both are representations of the (simply-typed) λ-calculus, and both are topological, but beyond that we see little resemblance between the two constructions. Our consistency conditions are axiomatic, and we use the topology to do logic and so to break λ-down into $\forall$ and $-\!\bullet$. Substitution σ is managed by axioms.

12.1.7. In what universe does this paper take place?. The points built in Theorem 6.1.13 do not have finite support, and in Definition 3.2.1 we assume a set with a permutation action but not necessarily a nominal set. Thus, this paper does not take place entirely in the topos of nominal sets; we do whatever is convenient to get the results we need and do not commit to any specific logic when we get them, even though our main results can be stated entirely in the nominal sets universe. In this we are being typical mathematicians, reasoning freely in English about informally but precisely specified mathematical objects.[44]

12.2. Future work

In Subsection 11.10 we noted that $points_{\Pi}$ has plenty of structure. It remains to explore that structure: $points_{\Pi}$ is a lattice and so contains a logic. We know this has interesting structure, investigated from Subsection 11.1. That does not exhaust the possibilities: Appendices B.2 and B.3 note that $points_{\Pi}$ also supports an existential quantifier. What is the full logic of $points_{\Pi}$ and how can it be used to investigate the λ-calculus?

Proposition 11.5.6 notes that $points_{\Pi}$ also has a σ-action, which we characterise in different ways in Subsection 11.4.2; the characterisation in Lemma 11.4.9 seems particularly appealing. As we note in the body of the paper, there is probably a general theory here: a way of, given a σ-action on $\mathcal{X}$, building a σ-action on the nominal powerset of $\mathcal{X}$. Such a theory was already undertaken in [Gab09b], where constructions were applied to models of Fraenkel-Mostowski set theory; thus generating a huge class of huge σ-algebras, since there are many models of FM sets and many sets in each model. The construction in Proposition 11.5.6 suggests the possibility of a cleaner and/or alternative development of similar ideas.[45]

We have used nominal lattices to give semantics to the λ-calculus. Part of our axiomatisation is the compatibility conditions of Figure 3, which include the axiom $(\bullet\sigma)$ (substitution commutes with application). If we relax this axiom, we get a meta-programming environment (because it allows functions to 'detect' atoms in their arguments); conversely $(\bullet\sigma)$ says that functions cannot do this. Meta-programming is a large field which has proven resistant so far even to precise categorisation. Generalisations of $\mathsf{inDi}\forall_{\bullet}$ without $(\bullet\sigma)$ might be one place to start looking for mathematical semantics.

On a related note, in [GG10] we noted that λ has a dual construction, $\forall a_1 \ldots a_n.(t-\!\bullet s)$, of *pattern-matching* (i.e. it applies to points in the 'pattern' t and outputs the same points in the pattern s). We

[44] Something similar happens in category theory when we talk about 'the category of all sets'; what does that live in? This is usually left unspecified, which is usually fine.

[45] ...and this is exciting. Most of this paper works by building various σ-algebras over relatively simple nominal algebraic structures like sets of points. What more could be achieved if we gave ourselves an entire mathematical foundation structure to play with?

suspect this might be the semantic analogue of Jay's *pattern calculus* [Jay04]. More generally, $\mathsf{inDi}\forall_{\bullet}$ and $points_{\Pi}$ are not just for the λ-calculus; they are rich and interesting environments, combining computational and logical structures, and much more. We have used them as a bridge between lattice-theory and λ-calculus. We would go so far as to suggest that for some people this bridge might be just as interesting as the λ-calculus itself.

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A. MORE ON FRESH-FINITE LIMITS

A.1. Axiomatisation of fresh-finite limits

A key definition in this paper has been Definition 4.4.3; this is 'poset-flavoured', in the sense that $\wedge$ and $\forall$ were characterised using fresh-finite limits (Definition 4.1.2).

It is interesting to ask whether fresh-finite limits can be rephrased in the syntax of nominal algebra, using equalities subject to freshness side-conditions.

This has implications, because if this can be done then $\mathsf{inDi}\forall$ and $\mathsf{inDi}\forall_{\bullet}$ are algebraic varieties, and satisfy the nominal HSP theorems from [Gab09a; Gab13].[46] This gives us off-the-shelf factorisation theorems similar to those considered in [MS10] (see especially Theorem 14), and in general, it is useful to know when a class of structures is an algebraic variety.

DEFINITION A.1.1. A **bounded lattice** in nominal sets is a tuple $\mathcal{L} = (|\mathcal{L}|, \cdot, \wedge, \vee, \bot, \top)$ where:

— $(|\mathcal{L}|, \cdot)$ is a nominal set which we may just write $\mathcal{L}$,
— $\bot \in |\mathcal{L}|$ and $\top \in |\mathcal{L}|$ are equivariant **bottom** and **top** elements,
— $\wedge, \vee : (\mathcal{L} \times \mathcal{L}) \to \mathcal{L}$ are equivariant functions, such that $\wedge$ and $\top$ form an idempotent monoid and $\vee$ and $\bot$ form an idempotent monoid,

$$\begin{array}{llll} (x\wedge y)\wedge z = x\wedge(y\wedge z) & x\wedge y = y\wedge x & x\wedge x = x & \\ (x\vee y)\vee z = x\vee(y\vee z) & x\vee y = y\vee x & x\vee x = x & x\vee\bot = x \end{array}$$

— and $\wedge$ and $\vee$ satisfy *absorption*

$$x\wedge(x\vee y) = x \qquad\qquad x\vee(x\wedge y) = x.$$

Here, x, y, z range over elements of $|\mathcal{L}|$.

[46] The nominal HSPA theorem states that every nominal algebra model is a subobject of a homomorphic image of a cartesian product of atoms-abstractions of free algebras (atoms-abstraction for nominal algebras is defined in [Gab09a], as is 'free algebra' and so on). In spirit, this is like the factorisation of natural numbers into primes or any other number of similar factorisation theorems. It is useful because it constrains the structure of models, and it is one of the applications of abstract algebraic techniques.

The result proved in [Gab09a] considered an untyped syntax, but we expect it to generalise unproblematically to the typed case, if necessary.

$$
\begin{array}{lrl}
(\forall\alpha) & b\#x \Rightarrow & \forall b.(b\ a)\cdot x = x \\
(\forall\wedge) & & \forall a.(x\wedge y) = (\forall a.x)\wedge(\forall a.y) \\
(\forall\vee) & a\#y \Rightarrow & \forall a.(x\vee y) = (\forall a.x)\vee y \\
(\forall\leq) & & \forall a.x \leq x
\end{array}
$$

Fig. 4: Nominal algebra axioms for $\forall$

A bounded lattice is a poset by taking $x \leq y$ to mean $x\wedge y = x$ or $x\vee y = y$ (the two conditions are provably equivalent). Definition A.1.1 is the usual definition of a bounded lattice, but over a nominal set; but we have not done anything with it yet.

Definition A.1.2 exploits the nominal set structure to algebraise the universal quantifier:

Definition A.1.2. Suppose $\mathcal{L}$ is a nominal poset.

A **(nominal) universal quantifier** $\forall$ on $\mathcal{L}$ is an equivariant map $\forall : (\mathbb{A} \times \mathcal{L}) \to \mathcal{L}$ satisfying the equalities $(\forall\alpha)$ to $(\forall\leq)$ in Figure 4.

Lemma A.1.3. *$a\#\forall a.x$.*

Proof. Choose fresh b (so $b\#x$). By Proposition 2.3.3 $a\#(b\ a)\cdot x$. It follows by Theorem 2.3.1 that $a\#\forall b.(b\ a)\cdot x$. By $(\forall\alpha)$ $\forall b.(b\ a)\cdot x = \forall a.x$. □

Proposition A.1.4. *Suppose $\mathcal{L}$ is a bounded lattice with $\forall$ and $x \in |\mathcal{L}|$. Then $a\#x$ implies $\forall a.x = x$*

Proof. We reason as follows:

$$\forall a.x = \forall a.(x\vee x) \stackrel{(\forall\vee)}{=} (\forall a.x)\vee x$$

So $x \leq \forall a.x$. Furthermore by $(\forall\leq)$ $\forall a.x \leq x$, and we are done. □

Corollary A.1.5. *Suppose $\mathcal{L}$ is a bounded lattice and suppose $\mathcal{L}$ has a nominal universal quantifier $\forall$. Then $\forall a.x$ is the $a\#$limit for $\{x\}$.*

Proof. By $(\forall\leq)$ $\forall a.x \leq x$ and by Lemma A.1.3 $a\#\forall a.x$.

Suppose $z \in |\mathcal{L}|$ and $a\#z$ and $z \leq x$. It follows using $(\forall\wedge)$ that $\forall a.z \leq \forall a.x$ and by Proposition A.1.4 $\forall a.z = z$. □

Proposition A.1.6. *The notion of a nominal distributive lattice with $\forall$ from Definition 4.4.3 is characterised in nominal algebra as:*

> *a bounded lattice (Definition A.1.1) with a nominal universal quantifier (Definition A.1.2) that is distributive (Definition 4.4.1) and has a compatible σ-action (Definition 4.3.1).*

Proof. By routine calculations. The interesting part is to check that the characterisation of $\forall a$ from Definition 4.1.2 as a fresh-finite limit coincides with the algebraic characteristaion of $\forall a$ in Figure 4. The meat of this is handled by Proposition A.1.4. □

A.2. Support interpolation

We return to Proposition 4.2.3, which stated that in a nominal poset $\mathcal{L}$ if $\bigwedge^{\#a}x$ exists then so does $\bigwedge^{\subseteq supp(x)\setminus\{a\}}x$ and they are equal (this mattered to us for proving Lemma 5.2.3 and so Proposition 5.2.6).

What can we say about the other way around? When does the existence of $\bigwedge^{\subseteq S}x$ imply the existence of $\bigwedge^{\#a}x$?

We can note Proposition A.2.3 and Remark A.2.4.

Lemma A.2.1. *$supp(\bigwedge^{\subseteq S}x) \subseteq S$.*

Proof. By Lemma 2.4.3 $supp(B) \subseteq S$. We use Theorem 2.3.1. □

DEFINITION A.2.2. Say $\mathcal{L}$ has **support interpolation** when if $x \leq y$ then there exists a z such that

$$x \leq z \quad \text{and} \quad z \leq y \qquad \text{and} \qquad supp(z) \subseteq supp(x) \cap supp(y).$$

(*Interpolation* is used here by analogy with the concept in logic [GM05]; what we are interpolating is the set of names.)

PROPOSITION A.2.3. *Suppose $\mathcal{L}$ has support interpolation and suppose $\bigwedge^{\subseteq S} x$ exists for all x and $S \subseteq \mathbb{A}$. Then $\bigwedge^{\#a} x$ exists for all x and a and is equal to $\bigwedge^{\subseteq supp(x)\setminus\{a\}} x$.*

Proof. Suppose all strict fresh-finite limits $\bigwedge^{\subseteq S} x$ exist. Consider some a and x.

— *We show that if $x' \leq x$ and $a\#x'$ then $x' \leq \bigwedge^{\subseteq supp(x)\setminus\{a\}} x$.*
Suppose $x' \leq x$ and $a\#x'$. By support interpolation there exists a z with $x' \leq z \leq x$ and $supp(z) \subseteq supp(x) \cap supp(x')$. By assumption $z \leq \bigwedge^{\subseteq supp(x)\setminus\{a\}} x$ and so $x' \leq \bigwedge^{\subseteq supp(x)\setminus\{a\}} x$.
— *We show that $\bigwedge^{\subseteq supp(x)\setminus\{a\}} x$ is least with this property.*
By Lemma A.2.1 $supp(\bigwedge^{\subseteq supp(x)\setminus\{a\}} x) \subseteq supp(x)\setminus\{a\}$. Thus in particular $a\#\bigwedge^{\subseteq supp(x)\setminus\{a\}} x$. Also by construction $\bigwedge^{\subseteq supp(x)\setminus\{a\}} x \leq x$.
Thus for any other greatest element y in $\{x' \mid x' \leq x \wedge a\#x'\}$, $\bigwedge^{\subseteq supp(x)\setminus\{a\}} x \leq y$.

Thus, $\bigwedge^{\#a} x$ exists and is equal to $\bigwedge^{\subseteq supp(x)\setminus\{a\}} x$. □

REMARK A.2.4. In the absence of support interpolation Proposition A.2.3 may fail. For instance, consider a nominal poset consisting of singleton atoms $\{a\}$ and unordered pairs of atoms $\{a, b\}$ and an element $*$, such that:

— $\{a, b\} \leq \{a\}$ for all (distinct) a and b.
— $* \leq \{a\}$ for all a.

We assume the natural pointwise permutation actions and $\pi \cdot * = *$, so that $supp(\{a, b\}) = \{a, b\}$ and $supp(\{a\}) = \{a\}$ and $supp(*) = \varnothing$.
Then $* = \bigwedge^{\subseteq\varnothing}\{a\}$ but $\bigwedge^{\#a}\{a\}$ does not exist since $* \not\leq \{a, b\}$.

B. ADDITIONAL PROPERTIES OF THE CANONICAL MODEL

We noted in Subsection 11.6 that $points_\Pi$ has structure above and beyond being in $\mathsf{inDi}\forall_\bullet$. We conclude with some further reflection on this.

These properties were not needed for our main results, but they seem striking enough to merit a note in an Appendix.

B.1. λ commutes with unions

We consider $\boldsymbol{\lambda}$ from Notation 10.2.1 in $points_\Pi$ and prove Corollary B.1.3: $\boldsymbol{\lambda}$ commutes with sets union (cf. a similar property for $\forall$ proved in Lemma 11.8.14).

This property is not valid in general in $\mathsf{inDi}\forall_\bullet$, because $-\!\!\bullet$ and $\forall$ do not commute with $\vee$ in general,[47] but it does hold in the canonical model $points_\Pi$.

LEMMA B.1.1. $q \circ p^\bullet = (q \bullet p)^\bullet$.

Proof. We reason as follows:

$$\begin{aligned} q \circ p^\bullet &= \bigcup\{q \circ p' \mid p \subseteq p'\} && \text{Definition 9.2.4} \\ &= \bigcup\{(q \bullet p')^\bullet \mid p \subseteq p'\} && \text{Definition 11.7.1} \\ &= (q \bullet p)^\bullet && \text{Fact of Def 11.7.1} \end{aligned}$$

□

LEMMA B.1.2. $p^\bullet -\!\!\bullet (\bigcup_i p_i^\bullet) = \bigcup_i (p^\bullet -\!\!\bullet p_i^\bullet)$.

[47] The closest we get to this in the general case is $(-\!\!\bullet\vee)$ from Figure 3 and $(\forall\vee)$ from Figure 4

Proof. We reason as follows:

$q \in p^\bullet \multimap\!\!\bullet (\bigcup_i p_i^\bullet)$	$\Leftrightarrow q \circ p^\bullet \subseteq \bigcup_i p_i^\bullet$	Proposition 9.2.6
	$\Leftrightarrow (q \bullet p)^\bullet \subseteq \bigcup_i p_i^\bullet$	Lemma B.1.1
	$\Leftrightarrow \forall r.(q \bullet p \subseteq r \Rightarrow \exists i.p_i \subseteq r)$	Definition 11.7.1
	$\Leftrightarrow \exists i.p_i \subseteq q \bullet p$	Suffices to take $r = q \bullet p$
	$\Leftrightarrow \exists i.(q \bullet p)^\bullet \subseteq p_i^\bullet$	Definition 11.7.1, fact
	$\Leftrightarrow \exists i.q \circ p \subseteq p_i^\bullet$	Definition 11.7.1
	$\Leftrightarrow \exists i.q \in p^\bullet \multimap\!\!\bullet p_i^\bullet$	Proposition 9.2.6
	$\Leftrightarrow p \in \bigcup_i (p^\bullet \multimap\!\!\bullet p_i^\bullet)$	Fact

$\square$

COROLLARY B.1.3. $\boldsymbol{\lambda} a. \bigcup_i p_i^\bullet = \bigcup_i \boldsymbol{\lambda} a.p_i^\bullet$.

Proof. We recall $\boldsymbol{\lambda}$ from Notation 10.2.1 and using Lemmas B.1.2 and 11.8.14 deduce that $\boldsymbol{\lambda} a. \bigcup_i p_i^\bullet = \bigcup_i \boldsymbol{\lambda} a.p_i^\bullet$. $\square$

B.2. An existential quantifier

$points_\Pi$ from Definition 11.1.3 has a universal quantifier, defined in Definition 11.3.1. Proposition 11.1.8 hints that an existential quantifier might also exist.[48] Indeed this is so, and it is not hard to construct:

DEFINITION B.2.1. Suppose $p \in |points_\Pi|$ and $A \subseteq \mathbb{A}$. Define $\exists a.p$ by:

$$\exists a.p = \{s \in |\mho| \mid \forall u \in |\mho|.s[a:=u] \in p\}$$

We shall see that $\exists a.p$ is dual to Definition 11.3.1.

LEMMA B.2.2. *If* $p \in |points_\Pi|$ *then* $\exists a.p \in |points_\Pi|$.

Proof. We verify the conditions of Definition 11.1.3:

(1) *Proof that* $s \in \exists a.p \land s \to_\Pi t$ *implies* $t \in \exists a.p$. Suppose $s \in \exists a.p$, so that $s[a:=u] \in p$ for every u. If $s \to_\Pi t$ then by condition 3 of Definition 10.3.7 also $s[a:=u] \to_\Pi t[a:=u]$, and using condition 1 of Definition 11.1.3 also $t[a:=u] \in p$. Thus $t \in \exists a.p$ as required.
(2) *Proof that* $\mathsf{И} b.b \,\sharp_\sigma\, \exists a.p$. Suppose b is fresh (so $b \# p$). Unpacking Definition 11.1.1, we need to show that $\forall s.(s \in \exists a.p \Rightarrow \forall v.s[b:=v] \in \exists a.p)$. Suppose $s \in \exists a.p$, so $\forall u.s[a:=u] \in p$. Since $b \# p$ by Proposition 11.1.6 also $b \,\sharp_\sigma\, p$ and $\forall u,v.s[a:=u][b:=v] \in p$. It follows that $\forall u,v.s[b:=v][a:=u] \in p$ thus $\forall v.s[b:=v] \in \exists a.p$, thus $b \,\sharp_\sigma\, \exists a.p$.

$\square$

LEMMA B.2.3. *If* $p \in |points_\Pi|$ *then* $\exists a.p \subseteq p$.

Proof. From Definition B.2.1 taking $u = a$ and using $(\sigma\mathbf{id})$ from Figure 1. $\square$

LEMMA B.2.4. *If* $p \in |points_\Pi|$ *then* $supp(\exists a.p) \subseteq supp(p) \setminus \{a\}$.

Proof. By Theorem 2.3.1 $supp(\exists a.p) \subseteq supp(p) \setminus \{a\}$ and by construction in Definition B.2.1 $a \,\sharp_\sigma\, \exists a.p$. We use Proposition 11.1.6. $\square$

PROPOSITION B.2.5. *If* $p, q \in |points_\Pi|$ *and* $q \subseteq p$ *and* $supp(q) \subseteq supp(p) \setminus \{a\}$ *then* $q \subseteq \exists a.p$.

In other words, $\exists a.p$ is the greatest point contained in p for which a is fresh; compare and contrast this with the dual characterisation of $\forall a.p$ in Definition 11.3.1.

[48] Why? Because the proof of Proposition 11.1.8 would be simple and direct if we assume that $\mathcal{P}$ is strictly finitely supported. The fact that it works for *all* finitely supported $\mathcal{P}$ suggests there might be some way of removing atoms from the support of $p \in \mathcal{P}$ while also making p smaller as a set. This is exactly what an existential quantifier on points would do (recall that everything is inverted/dual in $points_\Pi$).

Proof. Suppose $q \subseteq p$ and $s \in q$. It suffices to show that $s \in \exists a.p$. By Proposition 11.1.6 $a \,\sharp_\sigma\, q$, so that $s[a{:=}u] \in q$ for every $u \in |\mho|$. Since $q \subseteq p$, we also have that $s[a{:=}u] \in p$ for every u, and it follows by Definition B.2.1 that $s \in \exists a.p$. $\square$

REMARK B.2.6. Nominal spectral spaces do not have an existential quantifier in general. Given a nominal spectral space $\mathcal{T}$ and an open set $X \in \mathit{opens}(\mathcal{T})$, the obvious candidate for $\exists a.X$ is $\bigcup\{X[a{\mapsto}u] \mid u \in |\mathcal{T}^\partial|\}$. This is just the dual of Definition 5.2.1.

However, this is not necessarily an open set because $\bigcup\{X[a{\mapsto}u] \mid u \in |\mathcal{T}^\partial|\}$ is not necessarily *strictly* finitely supported. See condition 4 of Definition 7.1.1, and Remark 7.1.3.

B.3. Interaction of ∀ and ∃ with application and $\mho$

The universal and existential quantifiers interact with $\bullet$ similarly to how $\forall$ interacts with $\vee$. We need Lemma B.3.1 as a simple technical lemma:

LEMMA B.3.1. *If $p \subseteq p'$ then $p \bullet q \subseteq p' \bullet q$.*

Proof. Routine from Definition 11.3.1. $\square$

LEMMA B.3.2. *If $a\#q$ then $\forall a.(p\bullet q) \subseteq (\forall a.p)\bullet q$.*

Proof. By Theorem 2.3.1 and Lemma 11.5.10, $a\#(\forall a.p)\bullet q$. Thus by part 4 of Lemma 11.3.5, $\forall a.(p\bullet q) \subseteq (\forall a.p)\bullet q$ if and only if $p\bullet q \subseteq (\forall a.p)\bullet q$. By part 1 of Lemma 11.3.5 and Lemma B.3.1, $p\bullet q \subseteq (\forall a.p)\bullet q$ is true. $\square$

LEMMA B.3.3. *If $a\#q$ then $(\exists a.p)\bullet q \subseteq \exists a.(p\bullet q)$.*

Proof. Suppose $v \in (\exists a.p)\bullet q$. So there exist s' and t such that $\forall u.s'[a{:=}u] \in p \wedge t \in q \wedge s't \to_\Pi v$. In particular, $s't \to_\Pi v$. It follows by condition 3 of Definition 10.3.7 that $s'[a{:=}u](t[a{:=}u]) \to_\Pi v[a{:=}u]$ for every u. But $a\#q$ so by Proposition 11.1.6 $a \,\sharp_\sigma\, q$ so $t[a{:=}u] \in q$ for every u. Thus, $v \in \exists a.(p\bullet q)$. $\square$

LEMMA B.3.4. *Suppose $a\#u, c$. Then $(\exists a.q)[u{\leftarrow\!\!\dashv}c] = \exists a.(q[u{\leftarrow\!\!\dashv}c])$.*

Proof. Unpacking Definition B.2.1 and using Proposition 3.3.2, $s \in (\exists a.q)[u{\leftarrow\!\!\dashv}c]$ if and only if $\forall v.s[c{:=}u][a{:=}v] \in q$. Using $(\sigma\sigma)$ and $(\sigma\#)$ this is if and only if $\forall v.s[a{:=}v][c{:=}u] \in q$. By Proposition 3.3.2 again, this is if and only if $s \in \exists a.(q[u{\leftarrow\!\!\dashv}c])$. $\square$